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ABSTRACT

This is part one of a two-part SMSG mathematics text for elementary school students. One of the goals of the text is the development of mathematical ideas via appropriate experiences with things from the physical world and the immediate environment. The text materials provide an introduction to the study of mathematics in which growth is from the concrete to the abstract, from the specific to the general. The authors emphasize exploration and progressive refinement of ideas associated with both number and space. Chapter topics include: (1) sets of points; (2) addition and subtraction - review and extension; (3) describing points as numbers; and (4) arrays and multiplication. (MP)

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Mathematics for the Elementary School

Book 3

Teacher's Commentary, Part I

REVISED EDITION

Prepared under the supervision of the
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PREFACE

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught--at all levels, from the kindergarten through the graduate school.

With this in mind, mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). The general objective of SMSG is the improvement of the teaching of mathematics in grades K-12 in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge, and at the same time one which reflects recent advances in mathematics itself. Among the projects undertaken by SMSG has been that of enlisting a group of outstanding mathematicians, educators, and mathematics teachers to prepare a series of sample textbooks which would illustrate such an improved curriculum. This is one of the publications in that series.

The development of mathematical ideas among young children must be grounded in appropriate experiences with things from the physical world and the immediate environment. The text materials for grades K-3 provide for young children an introduction to the study of mathematics that reflects clearly this point of view, in which growth is from the concrete to the abstract, from the specific to the general. Major emphasis is given to the exploration and progressive refinement of ideas associated with both number and space.

These texts for grades K-3 were developed following the completion of texts for grades 4-6. The dynamic nature of SMSG permitted serious reconsideration of several crucial issues and resulted in some modification of earlier points of view. The texts for grades K-3 include approaches to mathematics which appear to be promising as well as approaches whose efficacy has been demonstrated through classroom use.

It is not intended that this book be regarded as the only definitive way of introducing good mathematics to children at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that this and other texts prepared by SMSG will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

Based on the teaching experience of elementary teachers in all parts of the country and the estimates of the authors of the revisions, it is suggested that teaching time be approximately as follows:

BOOK 3

Chapter	Time
1	3-4 weeks
2	8-10 weeks
3	2-3 weeks
4	1-2 weeks
5	4-6 weeks
6	4-5 weeks
7	1-2 weeks
8	3-4 weeks
9	1-2 weeks

Teachers are urged to try not to exceed these approximate time allotments so that pupils will not miss the important ideas contained in later chapters.

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Chapter I

SETS OF POINTS

Background

Introduction

This chapter is devoted to geometry. We shall study what may be called physical geometry--that is, the geometry of the world around us. The study involves a certain amount of abstraction, for the fundamental objects we shall deal with are not things we can pick up or feel or see. We shall think of a point, for example, as an exact location in space. A point, then, has no size or shape or color; it has no physical attributes at all except its location. We indicate a point by making a pencil dot or a chalk dot; but every child will agree that such a dot does not mark an exact location, and he will enjoy imagining the unseeable points.

We may remark that the geometry studied in college courses is of a higher degree of abstraction still. There the fundamental geometric objects like point and line are not defined at all, and the study proceeds deductively from certain formally stated assumptions about them (called axioms).

Our purpose here is to help the pupil observe and describe fundamental geometric relationships. The discussion is intuitive. In the primary grades we are not particularly concerned with formal deductions.

Several geometric concepts have already been introduced: point, line segment, line, ray, curve, closed curve, simple closed curve, polygon, vertex, triangle, rectangle, square, circle, angle, congruence, and others. We now explore them further.

Point

By a point we mean an exact location--for example, the exact spot at the corner of a room where two walls and the ceiling meet. We indicate points by drawing dots; but we realize that a pencil dot, no matter how small, gives only an approximate location, not an exact one. (In fact, it is clear that a pencil dot on a sheet of paper covers infinitely many points--that is, more than can be counted.) Nevertheless, in order to keep the language simple, we refer to the dots themselves as the actual points.

It is customary to denote points by capital letters.

A point is a fixed location: points do not move. The point at the corner of the ceiling remains even if the whole building falls down. Nevertheless, it must be remembered that fixing a location is a meaningful notion only with respect to some particular frame of reference. Frames of reference in common use are: the sun, the earth, a car, a person, a ruler. A point that is fixed with respect to one frame of reference need not be fixed with respect to a different one. For example, when a ruler is carried across the room, a point on the ruler remains fixed with respect to the ruler but does not remain fixed with respect to the earth.

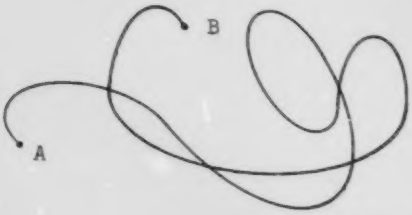
A geometric figure is any set of points.

Congruence

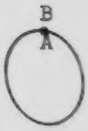
The idea of congruence in geometry is basic. Two geometric figures are said to be congruent provided that they have the same size and shape. A test is whether one will fit exactly on the other. In practice, the objects may not be conveniently movable; then one tests for congruence by making a movable copy of one and checking it against the other. Of course, all such tests, since they involve actual physical objects, often including the human eye, are only approximate. Nevertheless, in order to keep the language simple, we shall say, "The segments $\overline{AB}$ and $\overline{CD}$ are congruent" (rather than seem to be)--just as people say, "Johnny and Jimmy *are* exactly as tall as each other" (rather than *seem to be*).

Curve

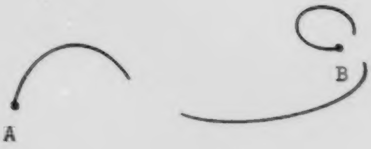
By a *curve* we mean any set of points followed in passing from a given point A to a given point B. Inherent in this definition is the intuitive notion of continuity; this is a curve:



and so is this:



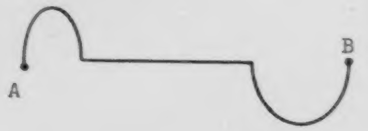
while this is *not* a curve:



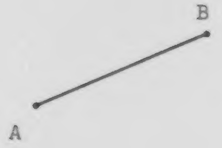
(However, it is a union of three curves.) We agree that a single point is not a curve.

It is also noteworthy that, according to the definition, a curve can be straight (in contrast with everyday usage).

This is a curve:



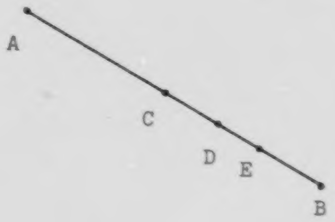
and so is this:



Line Segment

The last picture is an example of a line segment, that is, a straight curve. The endpoints are marked A and B; the line segment is denoted, accordingly, by either $\overline{AB}$ or $\overline{BA}$. Again, we agree that a single point is not a line segment.

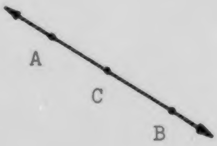
Observe that a line segment can always be expressed in many different ways as a union of other line segments. For example, the line segment $\overline{AB}$ shown here is the union of the line segments $\overline{AC}$ and $\overline{CB}$, the union of the line segments $\overline{AD}$, $\overline{AE}$, and $\overline{CB}$, etc.



Line

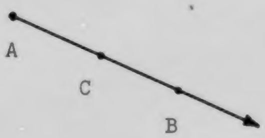
When a line segment is extended infinitely far in both directions, we get a line. Such extensions are only conceptual, of course, not practical. A line has no endpoints. No matter how far out we go in either direction along a line, still more of the line will lie ahead. The infinite extent is indicated by arrows.

The line containing points A and B is denoted by $\overleftrightarrow{AB}$. The line shown contains points A, B, and C; some names for this line are, therefore, $\overleftrightarrow{AB}$, $\overleftrightarrow{BA}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$, etc.

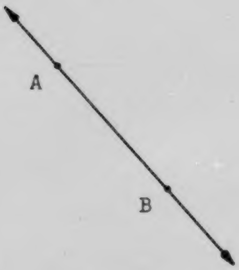


Note that, although $\overline{AB}$ and $\overline{AC}$ are different line segments, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are the same line.

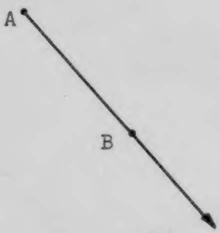
Just as a line is the infinite extension of a line segment in both directions a ray is the infinite extension of a line segment in one direction. A ray therefore has a single endpoint. The infinite extent of a ray is indicated by an arrow. The ray with endpoint A and containing another point B is denoted by $\overrightarrow{AB}$. The ray shown has endpoint A and contains points B and C; some names for this ray are, therefore, $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



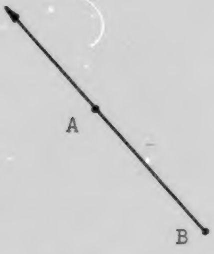
Note that, although $\overleftrightarrow{AB}$ and $\overleftrightarrow{BA}$ are the same line, $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are different rays:



$\overleftrightarrow{AB} = \overleftrightarrow{BA}$



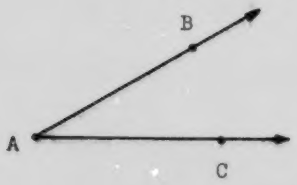
$\overrightarrow{AB}$



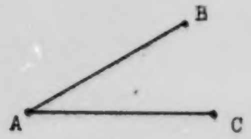
$\overrightarrow{BA}$

Angle

By an angle we mean the union of two rays having the same endpoint. (We exclude the case in which the two rays are part of the same line.) The common endpoint is called the vertex of the angle. The plural of "vertex" is "vertices". The angle formed by rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is denoted by $\angle BAC$ or $\angle CAB$.

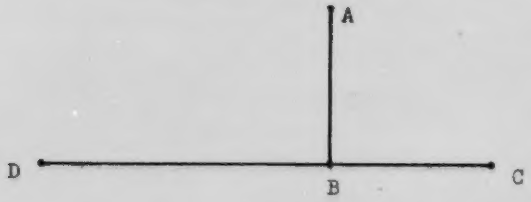


Two segments with a common endpoint determine an angle: segments $\overline{AB}$ and $\overline{AC}$ with common endpoint A determine the angle $\angle BAC$ with vertex A:

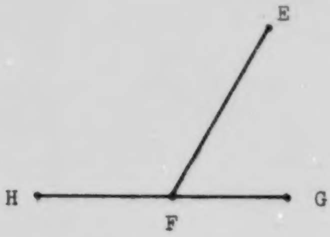


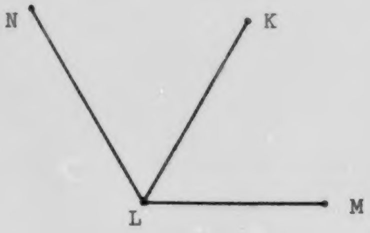
Right Angle

An angle is called a right angle if "two of them can fit together to form a line". In the diagram, $\angle ABC$ is congruent with $\angle ABD$, and the three points C, B, D lie on a line; therefore $\angle ABC$ and $\angle ABD$ are right angles.



Note that there are two parts to the definition: the part concerning congruence, and the part concerning the line. In the next diagram, $\angle EFG$ and $\angle EFH$ form a line but are not congruent, while $\angle KLM$ and $\angle KLN$ are congruent but do not form a line.



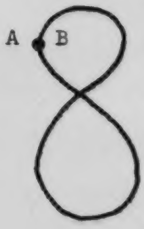


Plane

When a flat surface such as a table top, wall, or sheet of glass, or even this sheet of paper, is extended infinitely in all directions, we get a plane. Notice that if two points of a line lie in a given plane then the entire line is contained in the plane. Two intersecting lines determine a plane. In the teaching material, the infinite extent of the plane is not stressed.

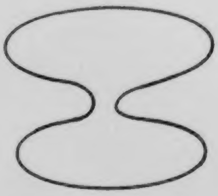
Closed Curve, Simple Closed Curve

We have called a curve any set of points followed in passing from a given point A to a given point B. When the points A and B coincide, the curve is said to be closed.



A closed curve

A closed curve that lies in a plane and does not cross itself is simple.

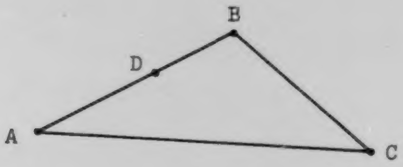


A simple closed curve

A simple closed curve has the interesting property of separating the rest of the plane into two subsets, an inside or interior (the subset of the plane enclosed by the curve) and the outside or exterior. Any curve connecting a point of the interior with a point of the exterior necessarily intersects the simple closed curve. (It may be of interest that this seemingly obvious fact is actually quite hard to prove.)

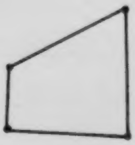
Polygon

An important class of simple closed curves is the class of polygons. A polygon is a simple closed curve that is a union of line segments. Recall that a line segment can always be expressed in many different ways as a union of line segments. Hence a polygon, too, can be expressed in different ways as a union of line segments.

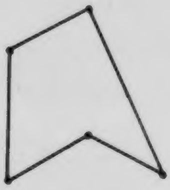


The union of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ =
the union of $\overline{AD}$, $\overline{DB}$, $\overline{BC}$, and $\overline{CA}$.

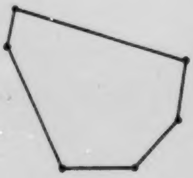
If we look at the various line segments in a polygon, we notice that they are of two kinds: those that are contained in other line segments, and those that are not contained in other line segments. For example, in the picture above, $\overline{AD}$ is of the first kind, since it is contained in the line segment $\overline{AB}$. On the other hand, $\overline{AB}$ is of the second kind, since it is not contained in any line segment except itself. Line segments of this second kind are called sides: a line segment in a polygon is called a *side* if it is not contained in any other line segment in the polygon. The polygon shown has three sides: $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$. A polygon of three sides is called a *triangle*. A polygon of four sides is a *quadrilateral*; of five sides, a *pentagon*; of six, a *hexagon*. (The last two names are not used in the teaching material.)



Quadrilateral



Pentagon



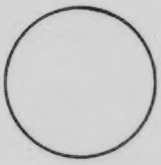
Hexagon

It may be observed that two consecutive sides of a polygon--that is, two sides with an endpoint in common--never lie on the same line. The endpoints of the sides are *vertices* (singular: *vertex*) of the polygon. The vertices of the triangle shown on page 8 are A, B, and C.

Rectangles are special kinds of quadrilaterals. Squares are special kinds of rectangles.

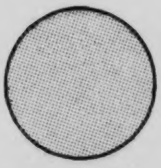
Region

The union of a simple closed curve and its interior is called a region. We refer to a triangular region, rectangular region, or circular region, etc., indicating that the simple closed curve is a triangle, rectangle, or circle, etc. For example, an ordinary sheet of paper is a rectangular region; the edges of the paper form a rectangle.



Circle

Interior



Circular region

As was noted above, the concepts discussed in this chapter have all had at least a rudimentary presentation in an earlier book. You should therefore feel free to modify the presentation according to the needs of the class. It is proper to omit or go lightly over sections that seem already familiar. It has been deemed better to provide a relatively complete presentation from which omissions may be made rather than to expect continued reference to material at a lower grade level.

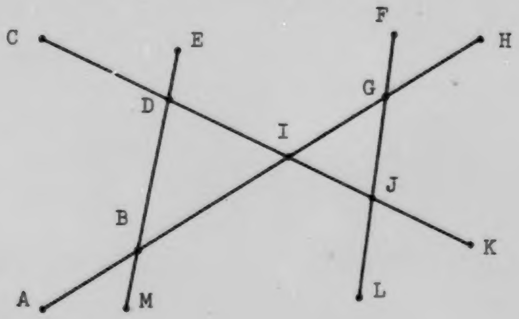
PUPIL'S BOOK

At the beginning of each geometry section you are encouraged to used physical models to illustrate geometric figures. On the pupil's book pages the phrase, "picture of", is frequently used in the first examples, then is deleted for ease of wording. Evaluate frequently how well the children distinguish between a picture of a figure and the figure itself, that is, the geometric idea.

The pupil's book pages may be used in a variety of ways depending on the needs of the class.

Diagnostic--the pages can be used independently to provide you with information concerning re-teaching.

Independent--the pages can provide independent review for children who seem to have ample understanding. As the work is completed, material of a more complex nature can be provided. The following example is an illustration of this kind of material.



How many points are named?

Try to name 20 different line segments.

Can you find ten more line segments?

Developmental--these pages can be illustrated from the flannel board, chalkboard, or overhead projector, then taught in a group or individually depending on need.

I-1. Points, curves, line segments

Objective: To review the ideas of a point, a curve, and a line segment.

Vocabulary: (Review) point, curve, line segment, subset, congruence, congruent.

Materials: String.

Suggested Procedure:

Use the playground and its equipment. You may wish to ask the class, before a play period, to look for line segments, curves, etc., outside; you may want to take the class outside for at least a part of the discussion; or you may ask questions after they have come in from a play period.

Points and Curves

To review the concept of point as an exact location, you may mention the points at various corners of the building, the point in the air to which a child can reach when he jumps, etc. Other examples are a spot on a ball, the point marked by a speck of dust, etc. Discuss what happens when the ball is kicked or thrown or when the speck of dust is blown away. Stress that sometimes we think of the point marked on the ball as remaining where it was even after the ball is kicked away, but at other times we like to think of the point as moving with the ball. The way we think about it depends on what we are interested in: the location on the ground or the location on the ball.

Any path children take on the playground is a curve. The path through which a ball travels is a curve.

It is hoped that children will arrive at the following ideas, which should then be used in their work.

a. A point is an exact location. We draw a dot and call it a point; but we know that the dot really marks many points, not just one.

b. We can think of a point as something that does not move or as something that does move. When we push a desk in the room, a point at a corner of the desk does not move on the desk but does move in the room.

c. Any set of points traced out in going from one point to another is called a curve. A curve can be straight.

Pupil's book, page 1: Points and curves

Ideas

A mark is used to show a point.

A curve is a set of points.

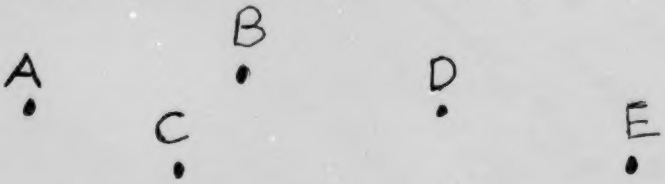
Examples 1, 2, and 3.

Some attention may need to be given to the first use of marks for points. You will find that these marks are usually designated by letters. To avoid stereotyping no common pattern is used.

Points and Curves

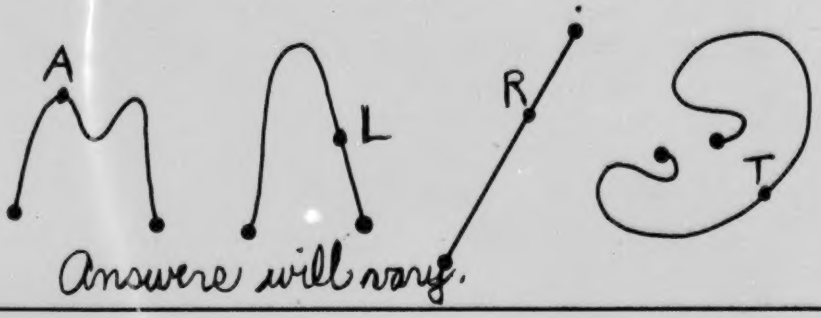
1. Mark five points below.

   Name them with the first five letters of the alphabet.



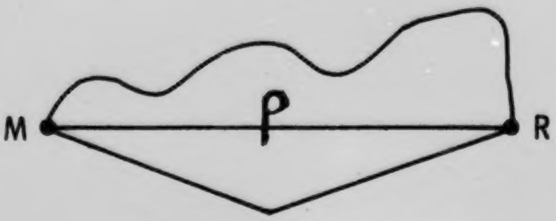
2. Mark a point on each curve.

   Name each point with a different letter of the alphabet.



Answers will vary.

3. Put the letter P on the picture of the straight curve from M to R.



* • Line segments

Examples of line segments may be the edge of a basketball court, the edge of a blackboard, a tightly stretched jump rope. Line segments are named by their endpoints, and names for the endpoints of the line segments noted might be given.

You may also wish to use the classroom itself, with line segments and other curves identified as the edge of a book, the edge of a sheet of paper, a drawing, the edge of a door, a yardstick, a crease made by the folding of a paper, a piece of string. Show how several line segments can have a common endpoint.

Congruence of line segments can be discussed by comparing the edges of two desk tops. Have the children compare them directly by pushing them together, and indirectly by means of a piece of string.

Some ideas to be gained are:

a. A curve can be straight. A straight curve from one point to another is called a line segment.

b. Many different line segments can have one endpoint in common.

c. There is only one line segment with two given endpoints.

d. When a line segment is fitted exactly onto a congruent line segment, the endpoints of the first coincide with the endpoints of the second, and the intermediate points of the first coincide with the intermediate points of the second.

The terminology of sets will probably not have been used to any great extent in the children's earlier discussions of geometry. At this stage it is hoped that you will gradually develop the ideas that a line segment is a set of points, every point of the line segment being a member of that set; that a line segment can be a subset of other line segments, rays, and lines; and so on.

* The dot is used throughout the text to indicate a change in activity or the development of another idea.

Pupil's book, pages 2-4: Line segments

Ideas--page 3

The straight curve from one point to another is a line segment.

A line segment has two endpoints.

A line segment may be named in two ways.

Example 1

The use of the symbol $\overline{\quad}$ is introduced.
The line segment $\overline{AB}$ can be written as $\overline{AB}$ or $\overline{BA}$.

Ideas--page 3

A line segment has two endpoints.
Many line segments may have a common endpoint.

Example 4

You should call attention to the meaning of $\overline{AF}$.

Ideas--page 4

Only one line segment can be drawn through two given endpoints.

A line segment contains many other line segments as subsets.

Example 6

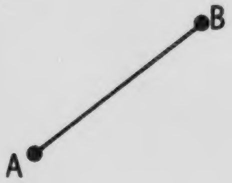
You may wish to remind the pupils that the endpoints A and B do not have thickness, even though the mark that is used to show them does.

Example 7

The pupils may have forgotten that a set is a subset of itself. $\overline{MN}$ can be listed as a subset of $\overline{MN}$.

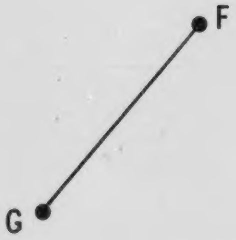
Line Segments

1. Here is a picture of a line segment.



Write a name for this line segment. $\overline{AB}$

2. Draw a line segment with F and G as endpoints.

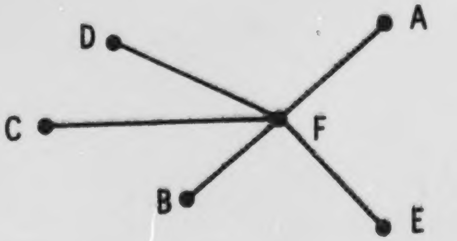


3. Write two names for the line segment above.

$\overline{FG}$ $\overline{GF}$

Line Segments

4. Here are some line segments that have point F as an endpoint.



One line segment is named below.

Name four other line segments.

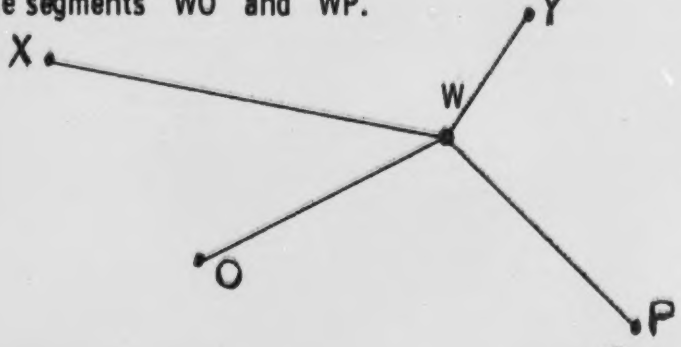
$\overline{AF}$ $\overline{EF}$ $\overline{BF}$ $\overline{CF}$ $\overline{DF}$

5. Draw two line segments that have point W as an endpoint.

Name these line segments $\overline{WX}$ and $\overline{WY}$.

Draw two more line segments that have W as an endpoint.

Name these line segments $\overline{WO}$ and $\overline{WP}$.

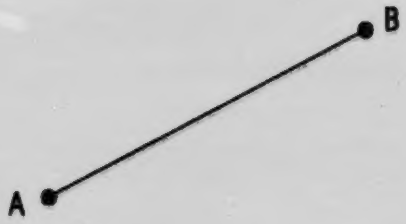


Can you draw more line segments with W as an endpoint? [✓ Yes] No

Line Segments

6. Below are two points, A and B.

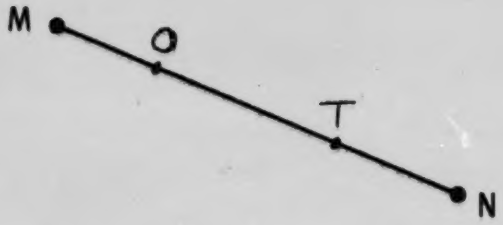
   Draw line segment $\overline{AB}$.



   How many line segments can you draw that have the two endpoints A and B? One

7. Here is line segment $\overline{MN}$.

   Mark two points on $\overline{MN}$. Name them O and T.



   $\overline{MO}$ is one subset of $\overline{MN}$.

   Name five other line segments that are subsets of line segment $\overline{MN}$.

   $\overline{MO}$ $\overline{MT}$ $\overline{MN}$ $\overline{OT}$ $\overline{ON}$ $\overline{TN}$

Pupil's book, pages 5-7: Congruence of line segments

Ideas--pages 5-7

Two line segments are congruent when one segment can be fitted exactly on the other.

Example 1

In looking at $\overline{AB}$ and $\overline{CD}$, children may not agree that these line segments are congruent. They may choose $\overline{EF}$ instead. Ask them how they could find out. A suggestion to measure the segments is quite likely. This answer is correct, but children should be helped to develop the idea that actual measurement is not necessary. The child need only make a copy of one segment, say by marking it off on the edge of a sheet of paper, and then try to fit this on the other segment. This is a direct application of the meaning of congruence.

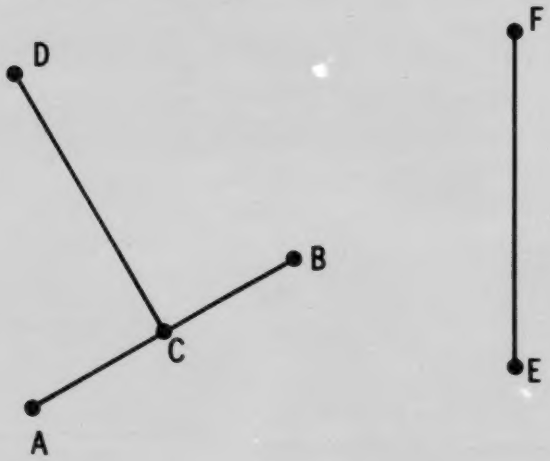
Example 2

On the question as to how to test for congruence, it may be suggested that one segment be cut out and compared. Lead from this to the idea of making a copy of one segment by tracing. Mention that it is often not feasible to bring two physical objects together for comparison. No one would dig up a tile from the floor to find if it were congruent to another tile.

Tracing paper may then be distributed. Suggest to the pupil that a very good way of making the tracing is to make a small dot at each endpoint and then join these dots. This copy may then be laid back on the segment to see if it is a good copy.

Congruence of Line Segments

1. Look at the segments below.



Do you think $\overline{AB}$ is congruent to $\overline{CD}$? Yes No

Answers will vary.

Compare $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ and show below what you find.

Make a ring around the right answer.

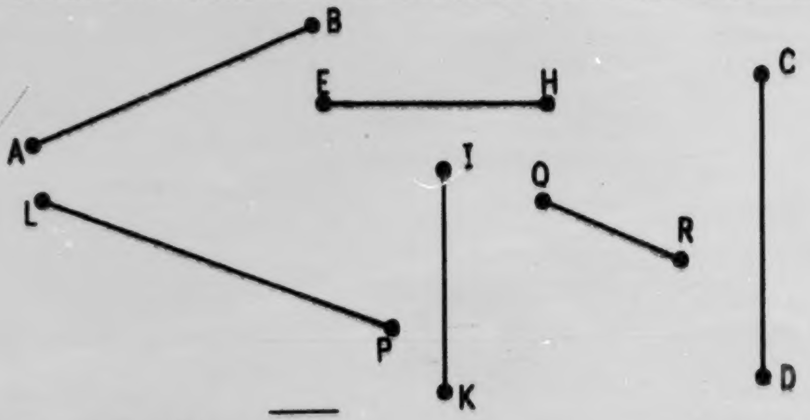
$\overline{AB}$ is congruent to $\overline{CD}$. [✓ Yes] No

$\overline{AB}$ is congruent to $\overline{EF}$. Yes [✓ No]

$\overline{CD}$ is congruent to $\overline{EF}$. Yes [✓ No]

Congruence of Line Segments

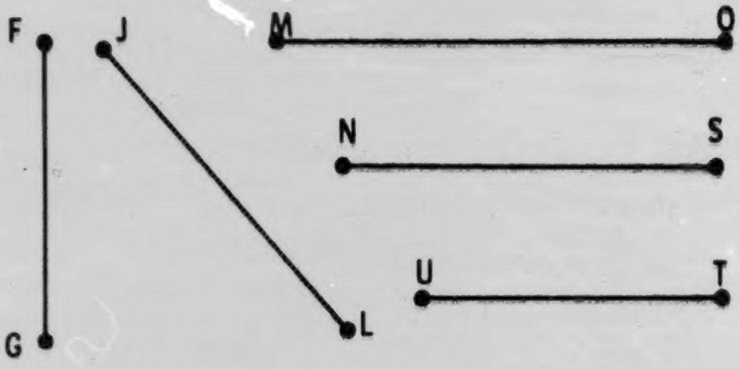
2.


A
B
E
H
C
D
I
K
L
P
Q
R

$\overline{AB}$ is congruent to $\overline{CD}$

$\overline{IK}$ is congruent to $\overline{EH}$

3.

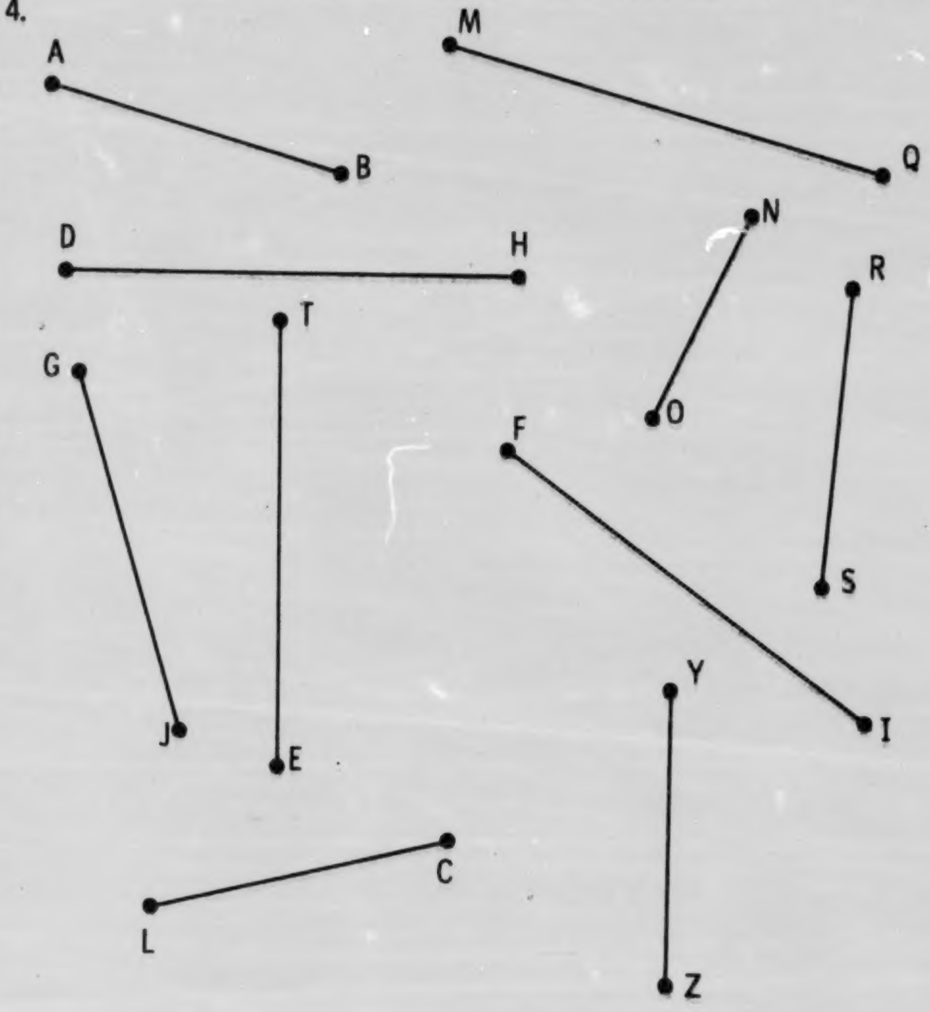

F
G
J
L
M
O
N
S
U
T

$\overline{FG}$ is congruent to $\overline{UT}$.

$\overline{JL}$ is congruent to $\overline{NS}$.

Congruence of Line Segments

4.


A
B
M
Q
D
H
N
O
R
S
T
E
G
J
F
I
Y
Z
C
L

$\overline{AB}$ is congruent to $\overline{RS}$, $\overline{YZ}$, and $\overline{LC}$.

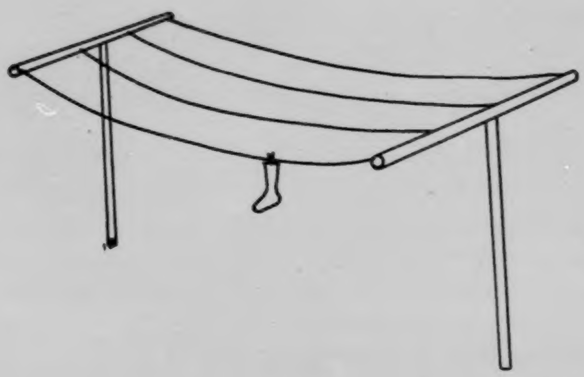
$\overline{MQ}$ is congruent to $\overline{FI}$, $\overline{TE}$, and $\overline{DH}$.

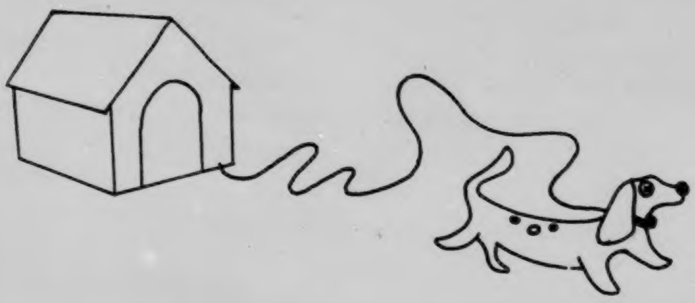
Pupil's book, page 8: Review

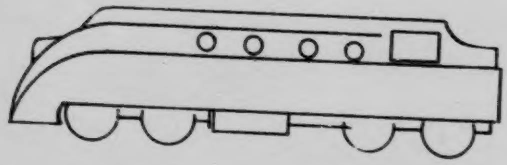
Page 8 of the pupil textbook is to be used orally. These statements and questions can serve as a asis for discussion. Feel free to extend or use other appropriate questions and statements. In using this page, have pupils identify the various pictures which appear.

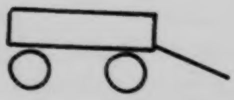
> This picture of a clothes line has many geometric figures represented in it. Name some. (Points, curves, line segments.)
>
> A sock is hanging on one of the clothes lines. What figure does that clothes line represent? (Curve.)
>
> What is the location of the clothes pin on the curve called (Point.)
>
> Find some pictures of several other curves on the page. (Outline of dog, leash, circles, line segments.)
>
> What kind of curve is represented by the top of the roof of the dog house? (Line segment.)
>
> Name several other line segments shown in the pictures. (Outline of wagon, dog house, braces, etc.)
>
> What line segments seem to be congruent? (On wagon, power pole, window of engine.)
>
> How could you test the segments for congruence? (Use a tracing.)
>
> Why does the clothes line not represent a line segment? Could it? How?
> (Not tight.) (Yes.) (Pull taut.)

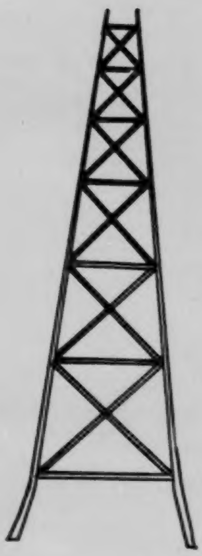
Review











I-2. Lines, rays, and angles

Objective: To review the ideas of a line, a ray, an angle, and a right angle.

Vocabulary: (Review) line, ray, angle, right angle, vertex (vertices), subset.

Materials: None.

Suggested Procedure:

Lines

The concept of *line* may be developed on the playground by thinking of one boundary of a basketball court as being extended on and on in both directions without end. Any other line segment may be thought of as being extended in the same way. Show how to write $\overleftrightarrow{AB}$ for the line through the two points A and B.

Ideas to be gained:

a. A line goes on and on in both directions without end.

b. There are many lines through any given point.

c. Through two given points there is only one line.

Pupil's book, pages 9-10: Lines

Ideas--page 9

A line is evolved through longer and longer line segments.

A line has no endpoints.

Lines contain line segments as subsets.

There are many names for the same line.

Any two points on a line name the line.

Example 1

You may prefer to use this as a group lesson by demonstrating the procedure of extending $\overline{AB}$ as $\overline{CD}$, as $\overline{EF}$, and as $\overleftrightarrow{EF}$, and having the pupils work with you.

Ideas--page 10

Many lines pass through any given point.
Only one line passes through two given points.

Example 3

There are four answers. The symbol $\overleftrightarrow{\quad}$ for line is introduced. Ask pupils to distinguish between $\overline{\quad}$ and $\overleftrightarrow{\quad}$.

Lines

1. Find the points E, C, A, B, D, and F named on the page.

Draw $\overline{AB}$.

Draw $\overline{CD}$. Is $\overline{AB}$ a subset of $\overline{CD}$? [✓ Yes] No

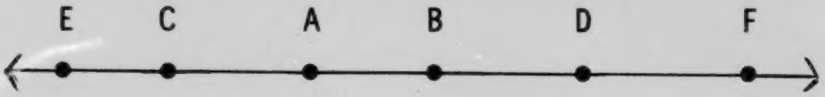
Draw $\overline{EF}$. The line segment $\overline{CD}$ is a subset of $\overline{EF}$.

Also $\overline{CD}$, $\overline{ED}$, $\overline{CF}$

Draw $\overleftrightarrow{EF}$. Is $\overline{EF}$ a subset of $\overleftrightarrow{EF}$? [✓ Yes] No

Is $\overline{EF}$ a subset of $\overline{EF}$? [✓ Yes] No

Can you show all of $\overleftrightarrow{EF}$? Yes [✓ No]



Some other names for $\overleftrightarrow{EF}$ are $\overleftrightarrow{CA}$, $\overleftrightarrow{AE}$, and $\overleftrightarrow{DF}$.

Write at least six other names below.

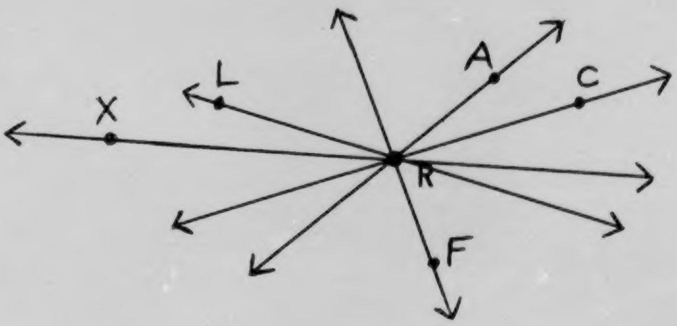
$\overleftrightarrow{EC}$ $\overleftrightarrow{EB}$ $\overleftrightarrow{ED}$ $\overleftrightarrow{CB}$ $\overleftrightarrow{CD}$ $\overleftrightarrow{CF}$

Lines

2. The point R is named below.

Draw five different lines through point R.

Mark and name another point on each line.



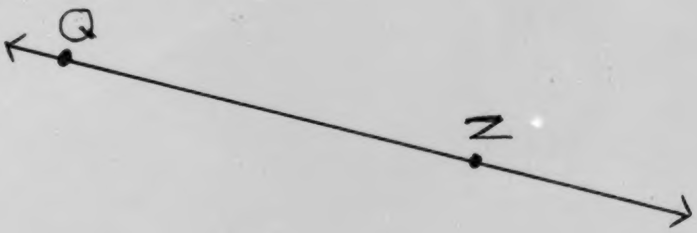
Name the lines you have drawn.

$\overleftrightarrow{AR}$ ______ $\overleftrightarrow{FR}$ ______ $\overleftrightarrow{CR}$ ______ $\overleftrightarrow{RL}$ ______ $\overleftrightarrow{XR}$ ______

Can many more lines be drawn through R? [✓ Yes] No

3. Mark two points below. Name them Q and Z.

Draw $\overleftrightarrow{QZ}$.



Can you draw a different line through Q and Z? Yes [✓ No]

• Rays

To review ray, have children choose a point, perhaps on the basketball court boundary, or at the tip of a pole, and imagine the set of points that goes one and on in one direction from that endpoint. Think also of the beam of a flashlight and of the rays of the sun. Show how to write $\overrightarrow{AB}$ for ray AB. Note that $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are not the same set of points.

Ideas to be gained:

a. Any point on a line separates the rest of the line into two subsets. Each of the subsets, together with that point, is called a ray.

b. Any two points on a line may be used to name the line.

c. In naming a ray, the endpoint is named first, and then any other point on the ray.

Pupil's book, pages 11-13: Rays

Ideas--page 11

A ray can be thought of as an extension of a segment in one direction.

The endpoint of a ray is named first.

Segments are subsets of rays.

Example 1

The symbol $\rightarrow$ for ray is introduced. Show that the endpoint is named first. Line segments are reviewed.

Example 2

To help children think of S as an endpoint, cover up the part of line $\overleftrightarrow{RS}$ that includes all points to the left of point S. Ask what ray the pupils see. ($\overrightarrow{ST}$.) Reverse the process to obtain $\overrightarrow{SR}$.

Ideas--page 12

Segments are subsets of rays.

Only two rays on a line can have a common endpoint.

Example 4

Encourage the pupils to use as much space as possible for their drawing. Help them see that a ray drawn across a corner of the space will necessarily be too small to note the points clearly.

Ideas--page 13

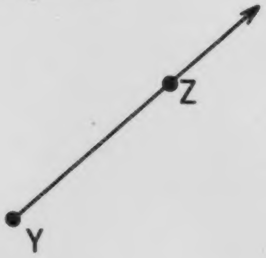
Many different rays can have a common endpoint.

Example 5

The letter T may be familiar to some as a symbol for the word true. Clarify that the letter is selected for this purpose in these examples.

Rays

1. Here is a picture of ray $\overrightarrow{YZ}$.



Name two points on $\overrightarrow{YZ}$. Y Z

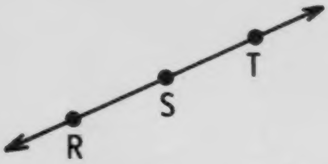
Name the endpoint of ray $\overrightarrow{YZ}$. Y

Is the endpoint named first? [✓ Yes] No

Name a line segment in the picture. $\overline{YZ}$

Does $\overrightarrow{YZ}$ go on from $\overline{YZ}$ in one direction only? [✓ Yes] No

2. Here is a picture of a line.



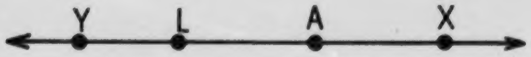
Name four rays on this line. $\overrightarrow{RT}$ $\overrightarrow{ST}$ $\overrightarrow{TR}$ $\overrightarrow{SR}$

Are the endpoints named first? [✓ Yes] No

Is $\overrightarrow{RT}$ another name for $\overrightarrow{TR}$? Yes [✓ No]

Rays

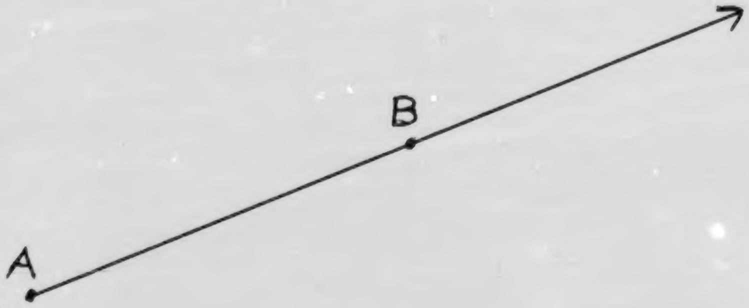
3. Here is another line.



How many rays on the line can have A as an endpoint? 2

Name three line segments on the line that have A as an endpoint. $\overline{AX}$ $\overline{AL}$ $\overline{AY}$

4. Draw a ray. Name it $\overrightarrow{AB}$.



Is $\overline{AB}$ a subset of $\overrightarrow{AB}$? [✓ Yes] No

Is $\overrightarrow{BA}$ another name for $\overrightarrow{AB}$? Yes [✓ No]

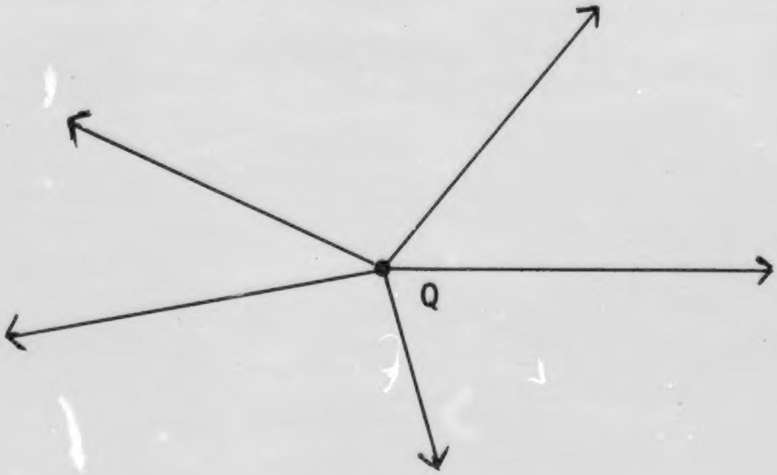
Rays

5. Mark the letter T as shown to complete each sentence correctly.

A line segment has
one endpoint ____
two endpoints T
no endpoints ____

A ray has	A line has
one endpoint T	one endpoint ____
two endpoints ____	two endpoints ____
no endpoints ____	no endpoints T

6. The point Q is marked below.



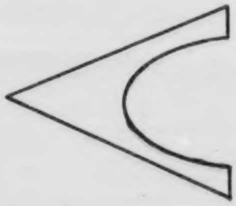
Draw five different rays above, each with endpoint Q.

- Angles

Many objects on the playground and in the classroom will suggest angles: the corners of the court, the braces of a backboard, the top of the slide, the hands of the clock. In each case identify the vertex. The concept that an angle is the union of two rays, rather than of two line segments, may be difficult and need not be belabored at this stage. According to the definition, the rays forming an angle do not lie on the same line; but again, this detail need not be belabored. Show the symbols $\angle BAC$ and $\angle CAB$.

Congruence of angles can be discussed with the help of two Judy clocks and the chalkboard compass. Set one clock at 8:00, say. Set the other at a different part of the clock; experiment with an angle much too small (e.g., at 1:10) and one much too large (e.g., at 12:30). Then try less obvious cases. When the decision is in doubt check with the compass: set it to the angle at one clock and hold it up for comparison against the other clock.

Other demonstrations can be made by means of tagboard cutouts; cut away a portion, as shown, so as to direct the child's attention to the angle rather than to a triangular region:



Demonstrate right angle by folding a sheet of paper as described on Pupil's book, page 21. (The first fold displays $\overline{AB}$. The second fold displays $\overline{CD}$; at the same time, it shows that $\angle ACD$ is congruent with $\angle BCD$.

Discuss right angles formed by the clock hands, the edges of the desk, the edges of an ordinary sheet of paper. Have the children put two sheets together or two desks together to form a line. In another test of congruence, each child can show how a sheet of paper fits a corner of his desk.

Ideas to be gained:

a. In naming an angle, the name of the vertex is always put in the middle.

b. An angle will fit exactly onto a congruent angle provided that the corners fit: the vertices coincide, and the rays forming the first angle extend in the same direction as the rays forming the second.

Pupil's book, pages 14-16: Angles

Ideas--page 14

An angle is formed by two rays with common endpoint.

In naming angles, the vertex is always in the middle.

An angle may be named in at least two ways.

Since rays may have many names, the angles formed by them may have many names.

Example 1

Before beginning this series the following should be developed orally with the pupils. Use overhead projector, flannel board, chalkboard, or other demonstration procedure.

Rays $\overrightarrow{AF}$ and $\overrightarrow{AD}$ form an angle. It is called angle FAD or angle DAF.

We can write this angle as $\angle$FAD or $\angle$DAF.

The point A is called the vertex of FAD.

It is the common endpoint of rays $\overrightarrow{AF}$ and $\overrightarrow{AD}$.

In naming an angle, the name of the vertex always goes in the middle.

Continue with angle QKR.

Help pupils select the point C in order to obtain more names for the angle.

Ideas--page 15

An angle is formed by two rays with a common endpoint.

In naming an angle, the vertex is always in the middle.

Ideas--page 16

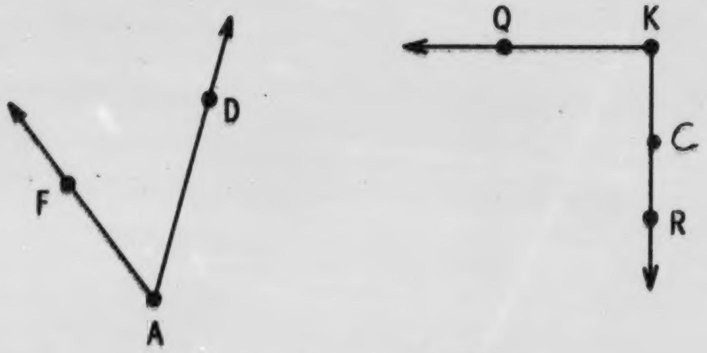
Since rays have many names, the angles formed by them may have many names.

Angles

1. Here are four rays.

The rays are named $\overrightarrow{AF}$, $\overrightarrow{AD}$, $\overrightarrow{KQ}$, and $\overrightarrow{KR}$.

These rays form two angles.



Name the two angles. $\angle$FAD $\angle$QKR

Give two other names for $\angle$FAD and $\angle$QKR. $\angle$DAF $\angle$RKQ

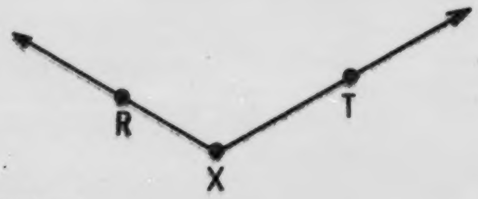
The vertex of $\angle$FAD is point A.

Name the vertex of the other angle. K

Mark a point C between K and R on ray $\overrightarrow{KR}$.

Now write two new names for $\angle$QKR. $\angle$CKQ $\angle$QKC

2. Here is another angle.

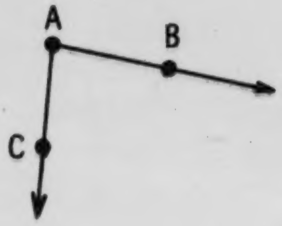


Name this angle. $\angle$RXT

Name its vertex. X

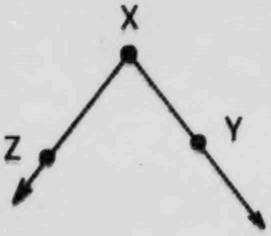
Angles

3. Name the vertex and the rays.



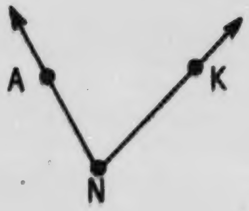
vertex A

rays $\overrightarrow{AC}$ $\overrightarrow{AB}$



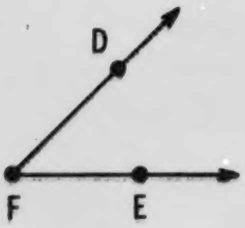
vertex X

rays $\overrightarrow{XZ}$ $\overrightarrow{XY}$



vertex N

rays $\overrightarrow{NA}$ $\overrightarrow{NK}$

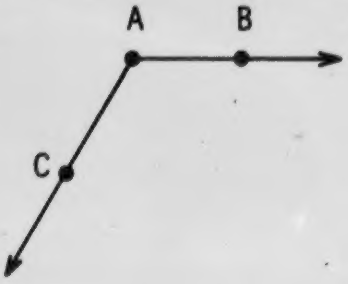


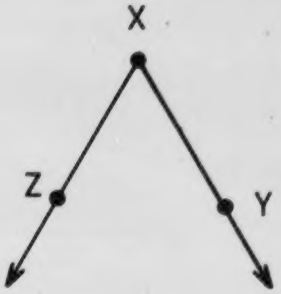
vertex F

rays $\overrightarrow{FD}$ $\overrightarrow{FE}$

Angles

4. Write two names for each angle.

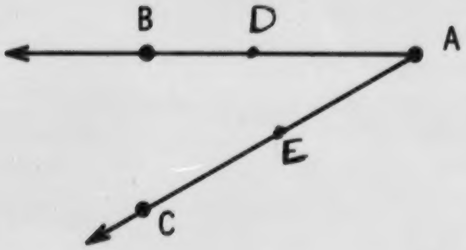




$\angle CAB$ $\angle BAC$

$\angle ZXY$ $\angle YXZ$

5. Below is a picture of $\angle BAC$.



Mark another point on $\overrightarrow{AB}$. Name it D.

Mark another point on $\overrightarrow{AC}$ and name it E.

Is $\overrightarrow{AB}$ the same ray as $\overrightarrow{AD}$? [✓ Yes] No

Is $\overrightarrow{AC}$ the same ray as $\overrightarrow{AE}$? [✓ Yes] No

Is $\angle BAC$ the same angle as $\angle EAD$? [✓ Yes] No

Is $\overline{BD}$ a subset of $\overrightarrow{AB}$? [✓ Yes] No

Pupil's book, pages 17-18: Congruence of Angles

Ideas--pages 17-18

Angles are congruent when the corner of one angle can be fitted on the corner of another.

Examples 1-5

Some help should be given the children in making a tracing of one angle and attempting to fit it onto another angle.

Pupil's book, pages 19-23: Right Angles and Congruence

Ideas--pages 19-23

When two congruent angles can be placed against each other with a ray in common so that the other two rays form a line, then the two angles are right angles.

A right angle can be formed by folding a sheet of paper in a particular way.

Example 1

Some children may point out that the angles are not congruent and no tracing is needed. Agree but ask them to check their observation.

Example 5

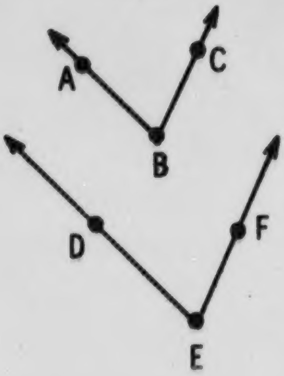
The use of this page should follow your demonstration on folding as shown on page 21. Construction paper 18" x 24" can be folded easily. Be sure the corners are jagged. Children may still need individual help as they proceed with the instructions on the page.

Example 6

Again the children are asked to follow several instructions. You may wish to read the page with the class and use it as an evaluative procedure with them.

Congruence of Angles

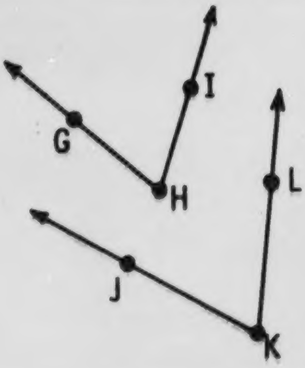
1.



Is ∠ABC congruent to ∠DEF?

[✓ Yes] No

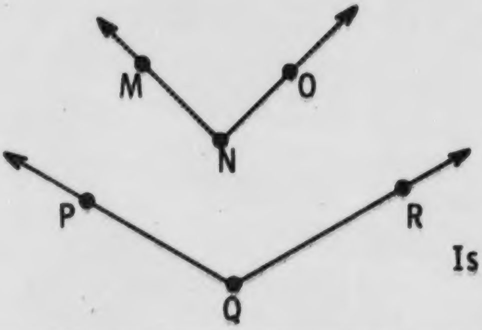
2.



Is ∠GHI congruent to ∠JKL?

[✓ Yes] No

3.

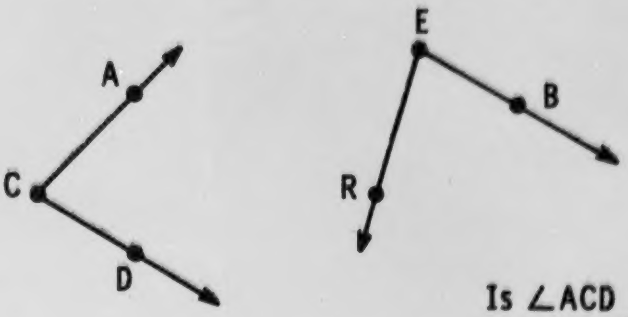


Is ∠PQR congruent to ∠MNO?

Yes [✓ No]

Congruence of Angles

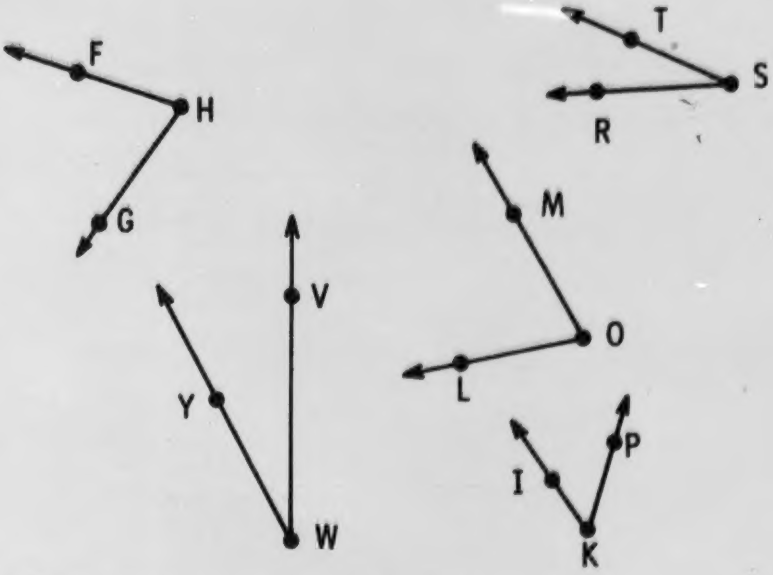
4.



Is ∠ACD congruent to ∠BER?

[✓ Yes] No

5.


F
H
G
T
S
R
M
O
L
V
Y
W
I
P
K

∠ FHG is congruent to ∠ MOL.

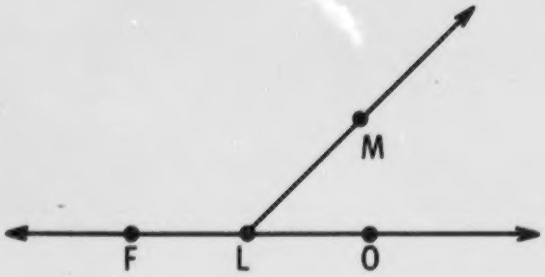
∠ YWV is congruent to ∠ TSR.

Right Angles and Congruence

1. The points F, L, and O lie on a line.

Make a tracing of one angle.

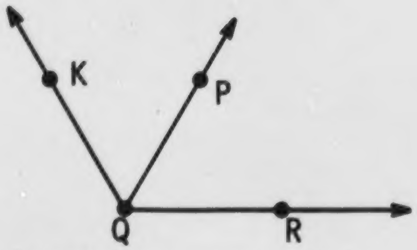
Test to see if the angles are congruent.



Are ∠MLO and ∠MLF congruent angles? Yes [✓ No]

Are ∠MLO and ∠MLF right angles? Yes [✓ No]

2. Test ∠PQR and ∠PQK to see if they are congruent.



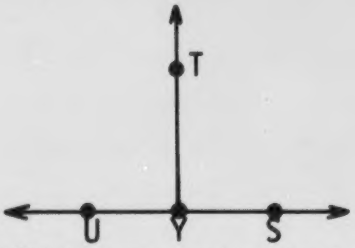
Are ∠PQR and ∠PQK congruent angles? [✓ Yes] No

Do the points K, Q, and R lie on a line? Yes [✓ No]

Are ∠PQR and ∠PQK right angles? Yes [✓ No]

Right Angles and Congruence

3. Test ∠TYS and ∠TYU to see if they are congruent.



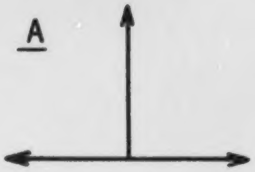
Are ∠TYS and ∠TYU congruent angles? [✓ Yes] No

Do the points S, Y, and U lie on a line? [✓ Yes] No

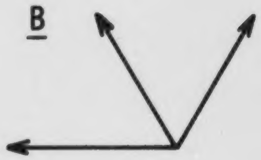
Are ∠TYS and ∠TYU right angles? [✓ Yes] No

4. Here are three pairs of angles; the pairs are called A, B, and C.

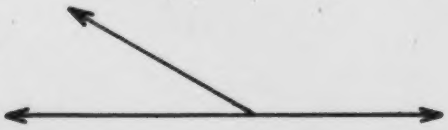
A



B



C



Set B is one set of congruent angles. Set A

Which other set looks like a pair of congruent angles? or A

Tell by looking which pair of angles could be right angles. A

Tell by testing which other pair of angles are congruent. B

Tell by testing which pair of angles are right angles. A

Forming a Right Angle

5. Here is one way to form a right angle.

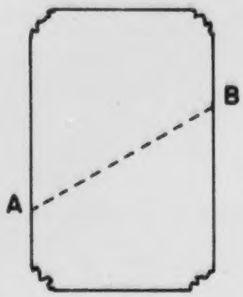
Step 1--Think about folding the sheet along $\overline{AB}$.

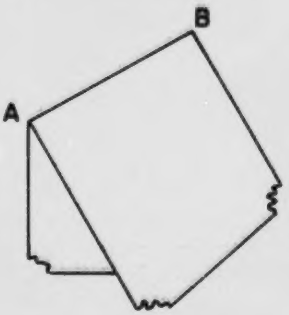
Step 2--Crease $\overline{AB}$ to show the line segment $\overline{AB}$.

Step 3--Think about folding the paper along $\overline{CD}$ so that endpoint B fits exactly on A.

Step 4--Crease $\overline{CD}$ to show the line segment $\overline{CD}$.

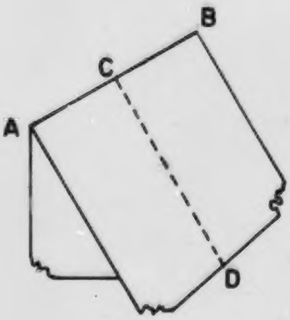
Step 1

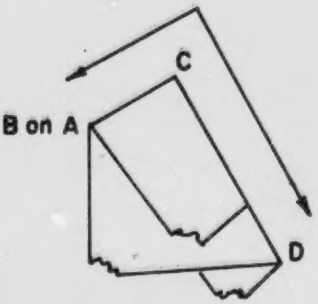




Step 2

Step 3





Step 4

Look at some of the curves and points we now have.

Segment $\overline{CA}$ is part of the ray $\overrightarrow{CA}$ with endpoint C.

Ray $\overrightarrow{CA}$ and ray $\overrightarrow{CD}$ form a right angle.

The vertex of the right angle is C.

Name the right angle. $\angle ACD$ or $\angle BCD$.

Right Angles

6. We can use our right angle to draw other right angles.

Below is ray $\overrightarrow{AB}$ with endpoint A.

Place the vertex of your right angle on point A.

Place one edge of your right angle along $\overrightarrow{AB}$.

Draw along the other edge.

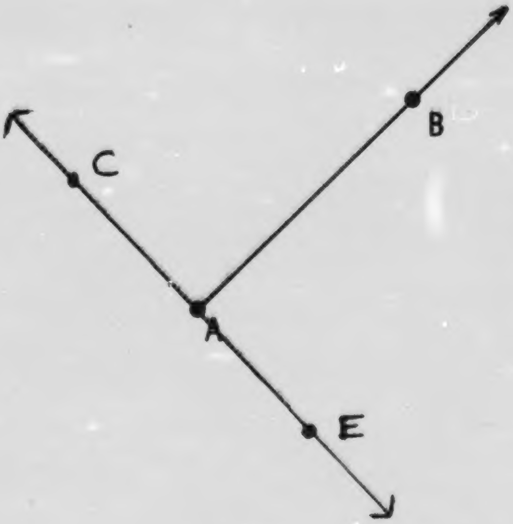
Name this ray. $\overrightarrow{AC}$ Names will vary.

Name this right angle. $\angle CAB$

Draw another right angle using $\overrightarrow{AB}$ and its endpoint A.

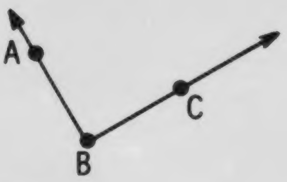
Name this angle. $\angle EAB$

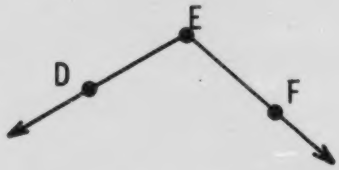
What kind of curve did you form with the two rays you drew? Line

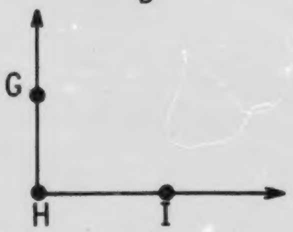


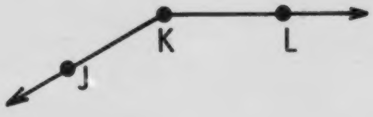
Right Angles

7. Test these angles to find the right angles.





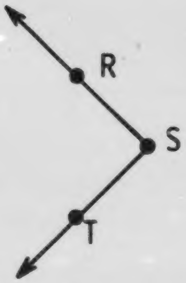


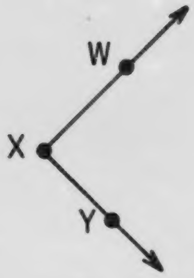


∠ABC and ∠GHI are right angles.

∠ABC is congruent to ∠GHI.

8. Use your angle to test if ∠RST and ∠WXY are right angles.





Is ∠RST congruent to ∠WXY? [✓ Yes] No

Do you think a right angle is always congruent to another right angle? [✓ Yes] No

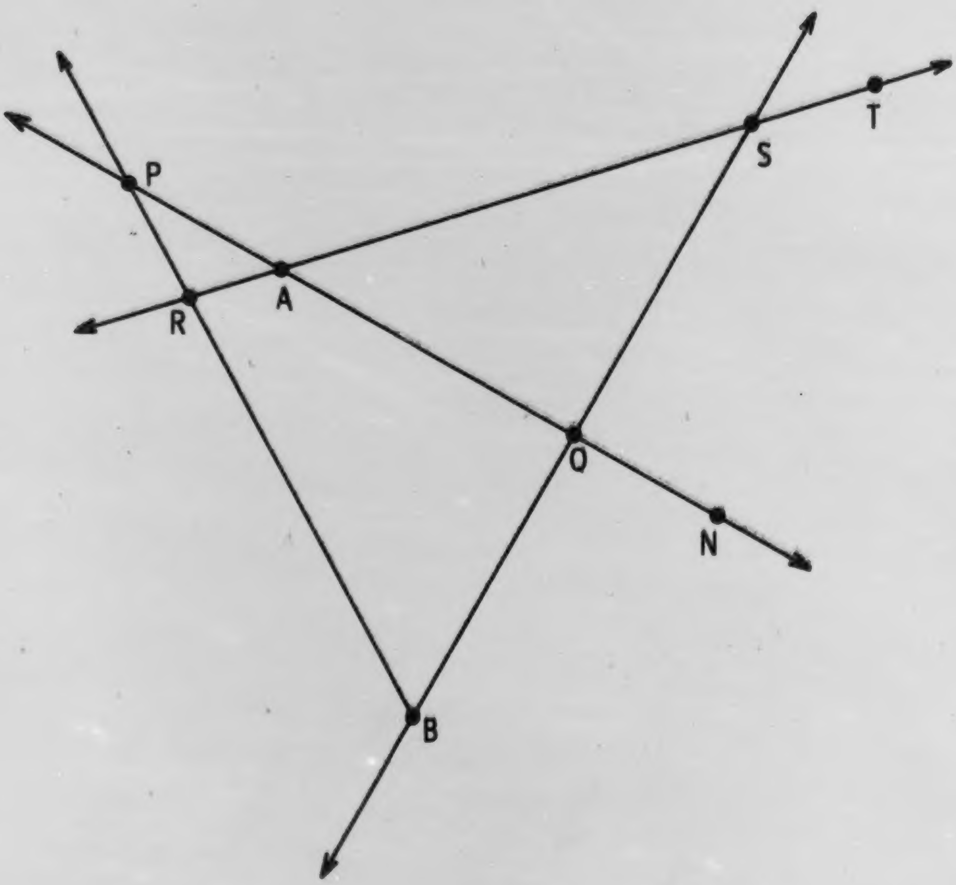
Pupil's book, page 24: Review

Much discussion and demonstration should precede the introduction of this page. It is a page to be done orally. Place illustrative pictures of the figures on the board.

The following statements and questions may aid in the discussion. You should amplify or change them according to the needs of your group.

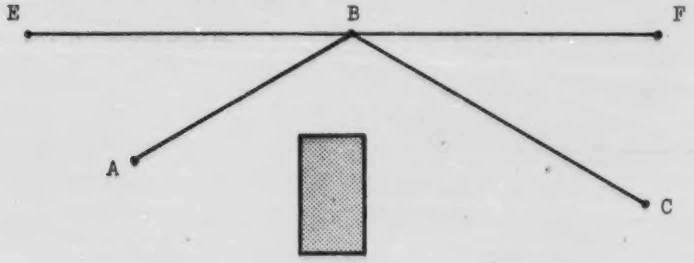
> On this review page we will talk about lines, rays, and angles.
>
> I have some pictures on the board. Let's see if we can describe each one.
>
> Is there a picture of a line? (Yes.)
>
> Look on your page. Can you find pictures of lines there? (Same for rays and angles.)
>
> At what point do lines RS and PQ meet? (A.)
>
> Which lines meet at point Q? ($\overleftrightarrow{BS}$, $\overleftrightarrow{PN}$.)
>
> Do any rays and lines meet? (Yes.)
>
> Name some points where rays and lines meet. (R, B, P A, Q, S.)
>
> Are there any line segments? (Yes.)
>
> Name some line segments. ($\overline{AR}$, $\overline{RB}$, $\overline{BQ}$, $\overline{QS}$, $\overline{AS}$, $\overline{PR}$, $\overline{AP}$, $\overline{QN}$, $\overline{ST}$.)
>
> Do you think any of the angles are congruent? (Yes.)
>
> How could you test these angles for congruence? (Make a tracing.)
>
> Are there any right angles? (Yes.)

Review



*★ Enrichment Activities

Experiment together with the children in rolling a ball against the wall at an angle and observing its path as it rolls back. In the diagram, the ball is sent from A, hits the wall at B, rolls back to C. The physical law is that the angle of incidence ($\angle ABE$) is congruent to the angle of reflection ($\angle CBF$).



Have a child hold his finger at A; have another child at the wall ready to hold his finger at B just after the ball goes by; and have another mark C. You can then draw a chalk line on the floor showing the path of the ball.

Repeat the experiment several times and lead the children to guess the physical law. In another problem, pick the points A and C in advance and then try to decide (either by trial and error or by the physical law) where the point B should be. In case there is an obstacle in the direct path from A to C (as in the diagram), the problem may take on added drama.

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* Starred sections are to be used on a differentiated basis with selected children rather than with all pupils.

I-3. Simple closed curves, polygons

Objective: To review the ideas of a closed curve, a simple closed curve, a polygon, a rectangle, and a square.

Vocabulary: Plane, side of a polygon; (Review) closed curve, simple closed curve, polygon, triangle, quadrilateral, rectangle, square.

Materials: Stovepipe wire.

Suggested Procedure:

Planes

Many flat surfaces can be found on the playground and in the classroom: basketball court, chalkboard, wall, ceiling, desk top, sheet of paper. The children may imagine extending such a flat surface to get a plane.

Simple closed curves

Closed curves will be found as painted boundaries on the basketball court, as borders of windows, and so on. Some of the closed curves will be simple; others will not be simple. Children may be asked to show, by walking, what the different terms mean. Confine the discussion to curves that lie in a plane. When a child finds a curve, you might ask him to tell what plane or flat surface the curve lies in.

Ideas to be gained:

a. A closed curve is a curve that starts and ends at the same point.

b. A simple closed curve is a closed curve that lies in a plane and does not cross itself.

Pupil's book, page 25: Closed Curves

Ideas

A closed curve is a curve that starts and ends at the same point.

Example 1

Attention should be called to the third figure in the rows three and four. Ask if these figures start and end at the same point.

Pupil's book, pages 26-27: Simple Closed Curves

Ideas

A simple closed curve is a closed curve that lies in a plane and does not cross itself. (The discussion of planes will remain informal in this book; flat surface as a plane will suffice.)

Example 2

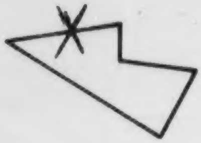
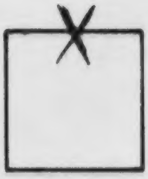
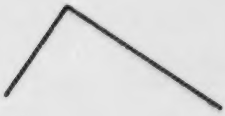
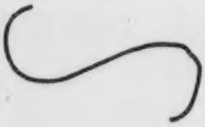
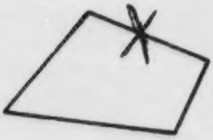
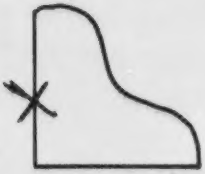
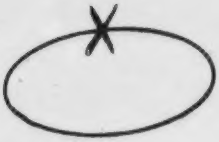
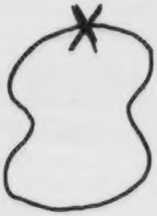
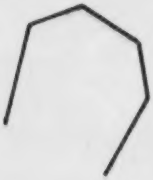
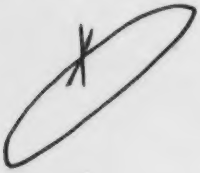
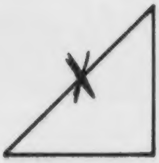
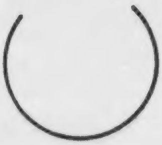
You may wish to demonstrate on the board that each non-simple, closed curve contains simple closed curves as subsets.

Example 3

You may use this as an optional example, as some pupils have difficulty seeing the closed curve. It should not be shaded in for presentation to the pupil. If he does this as an aid, permit it.

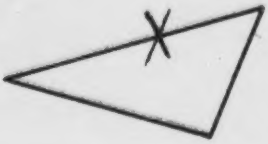
Closed Curves

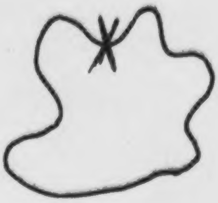
Mark an X on each closed curve.

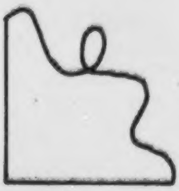













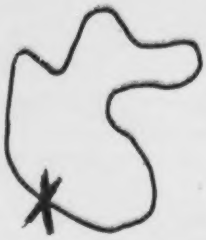
Simple Closed Curves

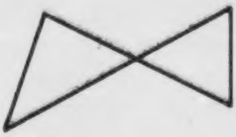
1. Mark an X on each simple closed curve.

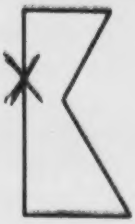




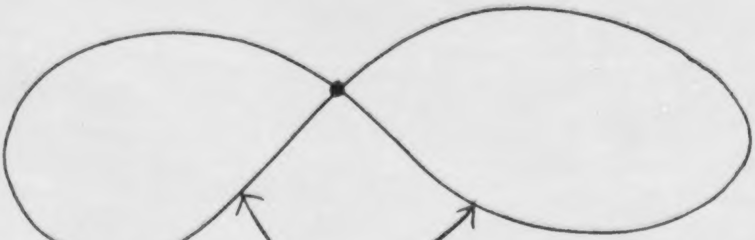








2. Draw a closed curve which is not simple.



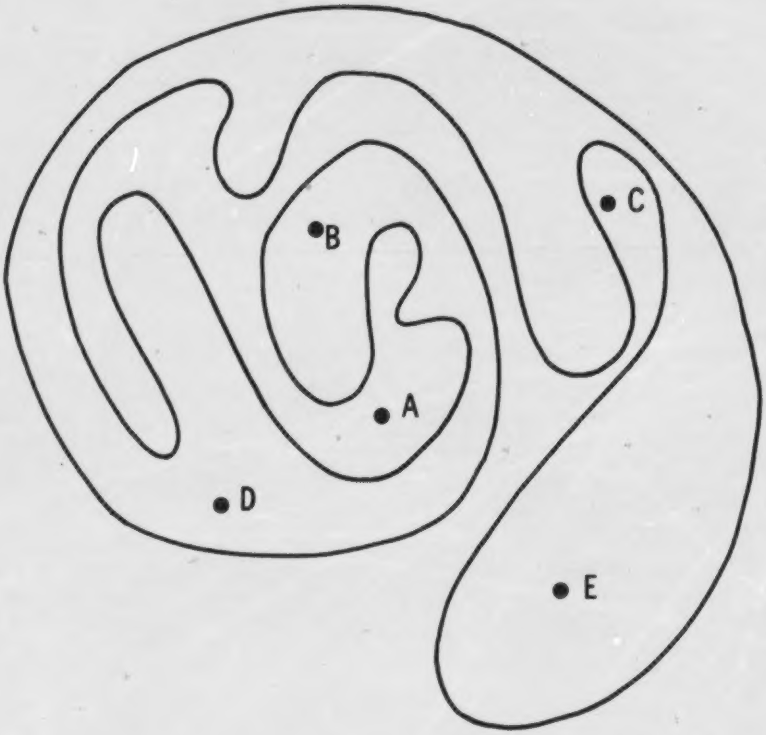
Color either curve.

Mark a point where this curve crosses itself.

Color a simple closed curve that is a subset of your curve.

Simple Closed Curves

3.



Is the curve a simple closed curve? [✓ Yes] No

Look at points A, B, C, D, E.

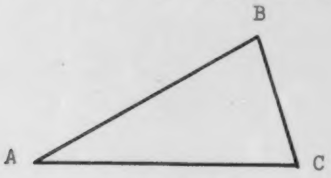
Write names of points which are inside the curve. B D E

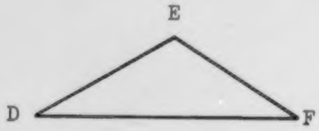
Write names of points which are outside the curve. A C

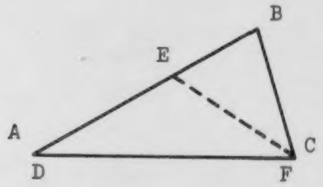
• Polygons

The terminology of sets is used again in the definition of *polygon*. Show polygons with different numbers of *sides*. Perhaps mention that "poly" means "many", that "gon" means "angle", that "tri" means "three", that "quadri" means "four", and that "lateral" means "side". Discuss whether a triangle can ever be congruent to a quadrilateral.

Use wire models to demonstrate congruence of triangles. Also, include two triangles which are not congruent but have a pair of congruent angles:







Rectangles

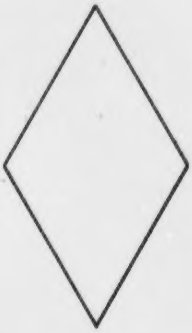
A *rectangle* is a quadrilateral whose four angles are all right angles. Perhaps mention that "rect" means "right" and refer to the word "correct" and its meaning. For a game, invent the word "correctangle". Have the children show congruence among all four angles of a rectangle by taking a sheet of paper and fitting each corner in turn with the same corner of the desk. Is there any other quadrilateral whose four angles are congruent? (No.)

Some children may wish to try to make one. Have the children discover that a rectangle has two pairs of congruent sides. Is there any other quadrilateral with two pairs of congruent sides? (Yes, any parallelogram.) Can a rectangle be congruent with another quadrilateral that is not a rectangle? (No.)

Can exactly two of the angles of a quadrilateral be right angles? (Yes. Give examples.) Exactly three? (No. If three then all four.)

A special kind of rectangle

A square is a rectangle whose four sides are all congruent. Is there any other quadrilateral with four congruent sides? (Yes, a "diamond". Perhaps the children can make some with pipe cleaners or sticks.)



Can a square be congruent with another quadrilateral that is not a square? (No.) Can a square be congruent with another rectangle that is not a square? (No.)

Ideas to be gained:

a. A polygon is a simple closed curve that is a union of line segments.

b. Some quadrilateral have special names.

c. Two polygons are congruent when their sides can be fitted exactly.

Pupil's book, page 28: Polygons

Ideas

A polygon is a simple closed curve that is a union of line segments.

Some quadrilaterals have special names.

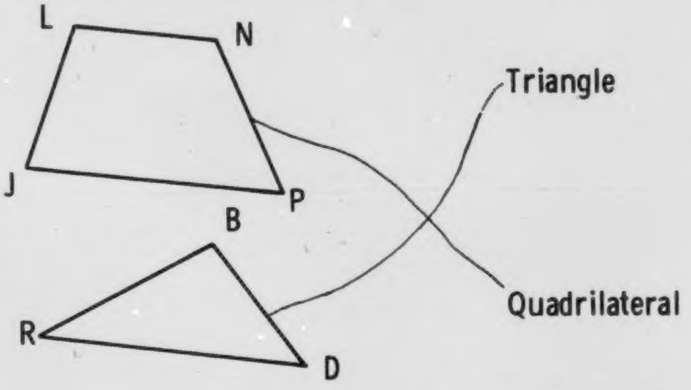
Examples 1 and 2

You may need to remind your class how you wish the names of the figures connected with the figures. Some children may connect the square to the word "rectangle". If so, further discussion on special names will be necessary.

Polygons

1. Here are pictures of different polygons.

Use your pencil to connect each polygon with its name.


L
N
J
P
B
R
D
Triangle
Quadrilateral

Name three line segments on the triangle. $\overline{RB}$ $\overline{BD}$ $\overline{RD}$

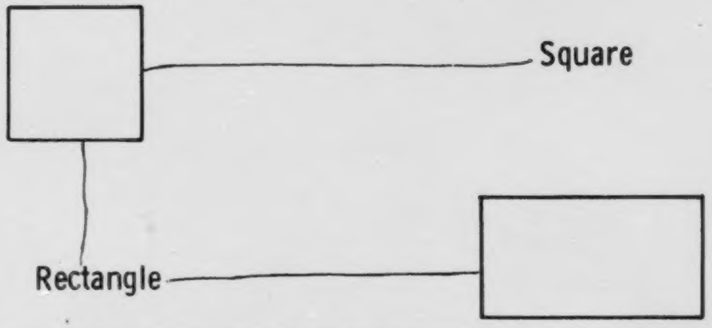
Name each vertex of the triangle. R B D

Name the sides of the quadrilateral. $\overline{JL}$ $\overline{LN}$ $\overline{NP}$ $\overline{PJ}$

Name each vertex of the quadrilateral. J L N P

2. Two quadrilaterals are shown below.

Connect them with their special names.


Square
Rectangle

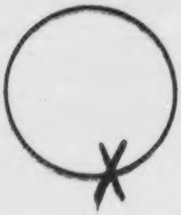
Pupil's book, page 29: Review

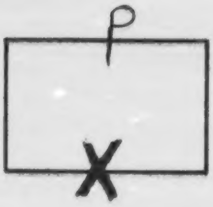
This pupil page follows the same pattern of pages that ends a review section. It is to be used orally. The following statements and questions are suggested:

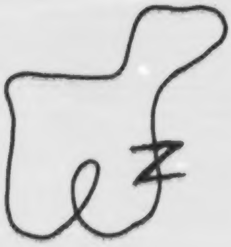
> How do we describe a simple closed curve? (A curve that starts and ends at the same point without crossing itself.)
>
> The page shows 12 curves. How many of these are simple closed curves? (6.) Mark them with an X.
>
> On the page are there any simple curves that are not closed? (Yes.)
>
> Mark these curves with a Y.
>
> What is the difference between a simple curve that is closed and one that is not closed? (Those not closed do not return to the same point.)
>
> There are other curves on the page that are not simple.
>
> How would you describe a curve that is not simple? (Crosses itself.)
>
> Find these curves and mark them with a Z.
>
> There are several simple closed curves that are unions of line segments. How do we name these curves? (Polygons.)
>
> There are four polygons. Mark with a P.
>
> In the second row the third curve has three sides.
>
> What name do we give to this curve? (Triangle.)

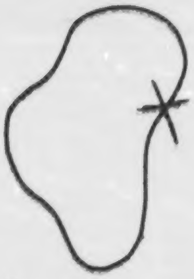
Continue in like manner with quadrilateral, rectangle, and square.

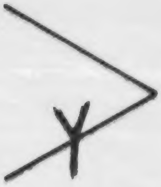
Review

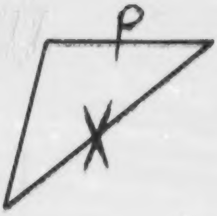


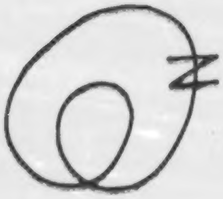


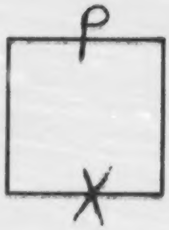


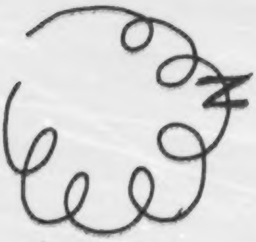


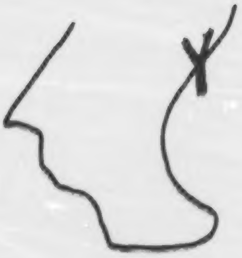


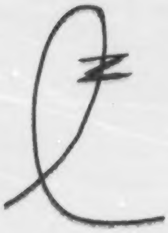


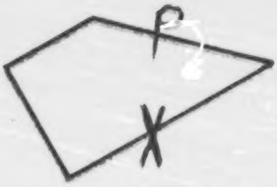












Pupil's book, pages 30-32: Congruence of Polygons

Ideas

Polygons are congruent when their sides fit exactly. Then their angles fit exactly.

Example 2

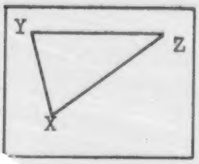
Remind the children that only the corners of the angles of the quadrilaterals are shown.

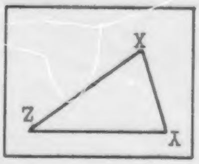
Example 3

The symbol △ is introduced for triangle. Triangle (△) XYZ will have to be turned to make a fitting. Stress the need to label the tracing in order to list the congruent line segments.

Example 5

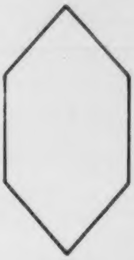
This example shows congruent figures in different positions. The congruence can be noted by observation.

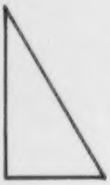




Congruence of Polygons

1. These simple closed curves are unions of line segments.





These kinds of curves are called Polygons.

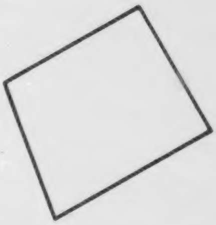
Can any of these polygons fit on each other exactly? Yes [✓ No]

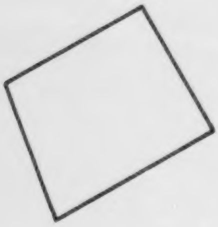
Do you think polygons can be congruent when they do not have the same number of sides? Yes [✓ No]

2. Two quadrilaterals are shown here.

Make a tracing of one curve.

Test to see if the sides and angles of the tracing fit exactly on the sides and angles of the other curve.





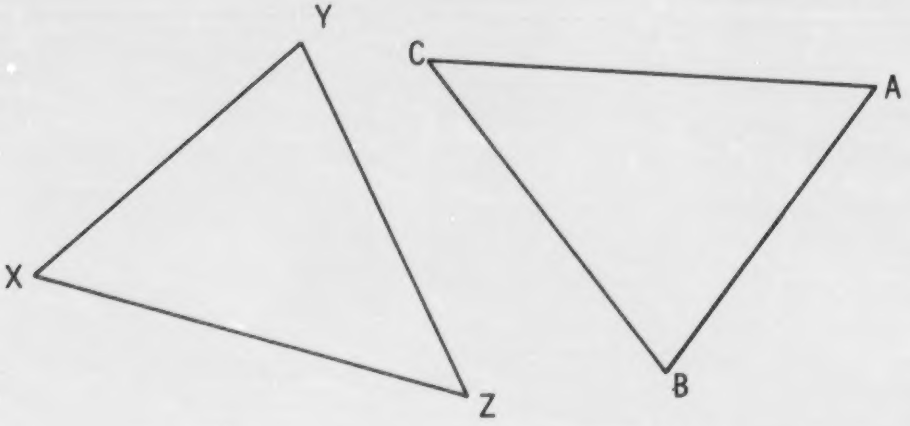
Do the sides fit exactly? [✓ Yes] No

Do the angles fit exactly? [✓ Yes] No

Are the curves congruent? [✓ Yes] No

Congruence of Polygons

3. These triangles are named $\triangle XYZ$ and $\triangle ABC$.



Do you think the triangles are congruent? [✓ Yes] No

Make a tracing of $\triangle XYZ$.

Mark the points X, Y, Z on the tracing.

Can you fit the tracing of $\triangle XYZ$ on $\triangle ABC$ without turning it? Yes [✓ No]

If you turn the tracing, can it fit on $\triangle ABC$? [✓ Yes] No

Line segment $\overline{XY}$ is congruent to $\overline{AB}$.

Line segment $\overline{YZ}$ is congruent to $\overline{BC}$.

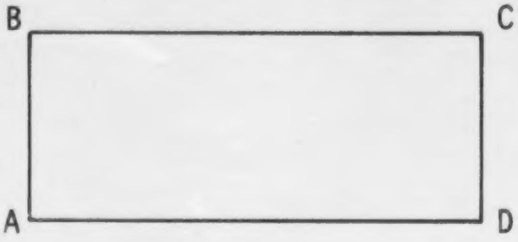
Line segment $\overline{XZ}$ is congruent to $\overline{AC}$.

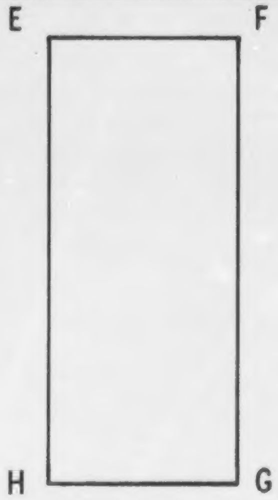
Is $\triangle XYZ$ congruent to $\triangle ABC$? [✓ Yes] No

Congruence of Polygons

4. Here are two rectangles.

We will call one rectangle ABCD.





Make a tracing of ABCD.

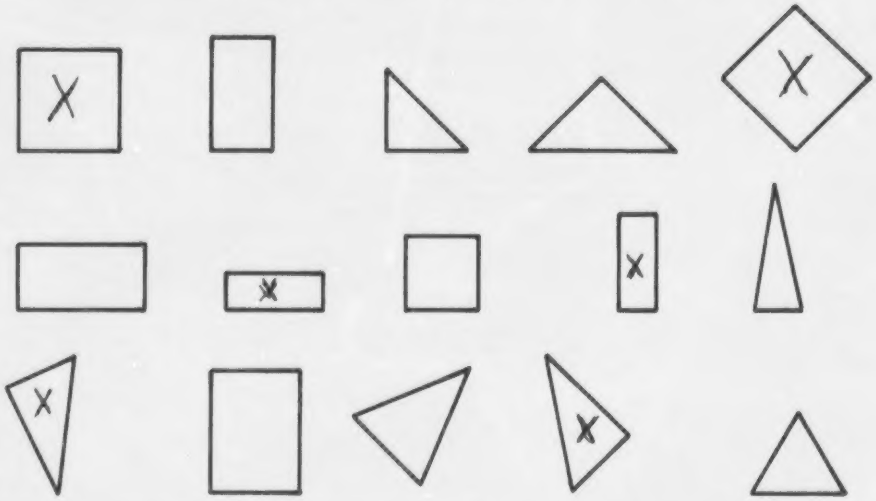
Test to see if the tracing fits exactly on EFGH.

Do the line segments fit exactly? [✓ Yes] No

Do you need to test the angles for congruence? Yes [✓ No]

Is ABCD congruent to EFGH? [✓ Yes] No

5. Put a cross in the two congruent figures in each row.



I-4. Regions

Objective: To review the idea of a region.

Vocabulary: (Review) region, interior, exterior.

Materials: Colored chalk; two congruent paper triangular regions and two congruent paper rectangular regions, all with the borders colored so as to stand out.

Suggested Procedure:

Chalkboard drawings can be used to emphasize the ideas of interior, exterior, and region. Use chalk of one color for the curve itself and a different color for the interior to clarify the terms.

Using the paper figures, demonstrate that when two triangular regions are congruent, then the triangles enclosing them are also congruent; likewise, when two triangles are congruent, then the regions they enclose are also congruent. Do the same with rectangles, etc. Discuss whether a triangular region can be congruent to a rectangular one.

Ideas to be gained:

a. Every simple closed curve has an interior (the subset of the plane enclosed by the curve) and an exterior (the subset of the plane outside the curve).

b. Any curve connecting a point in the interior of a simple closed curve to a point in the exterior necessarily crosses the simple closed curve.

c. A region is the union of a simple closed curve and its interior.

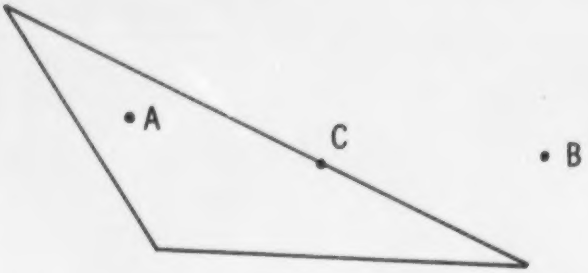
Pupil's book, page 33: Inside, On, and Outside

Ideas

Points can be located inside, on, and outside a simple closed curve.

Inside, On, and Outside

1. A polygon with three sides is called a triangle.

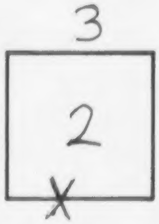


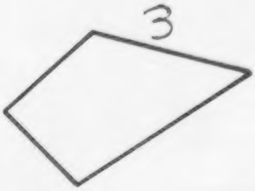
Name a point inside the triangle. A

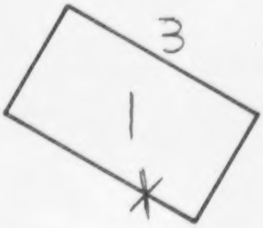
Name a point on the triangle. C

Name a point outside the triangle. B

2. Polygons with four sides are called quadrilaterals.







Write 1 in the rectangle that is not a square.

Write 2 inside the square.

Write 3 just outside each quadrilateral.

Mark X on each rectangle.

Pupil's book, pages 34-35: Interior and Exterior

Ideas

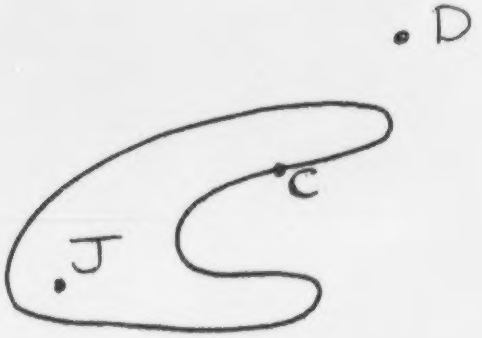
Every simple closed curve has an interior and exterior. Any curve connecting a point in the interior of a simple closed curve to a point in the exterior crosses the simple closed curve.

Example 3

You may wish to do part of this example as a class exercise until the pupils each draw a curve from A to B and agree it crosses the figure.

Interior and Exterior

1.

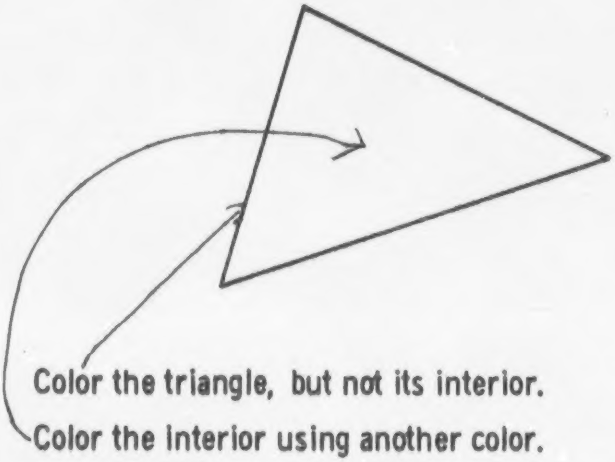


Mark a point J in the interior of this curve.

Mark a point C on the curve.

Mark a point D in the exterior of the curve.

2. Here is a triangle.



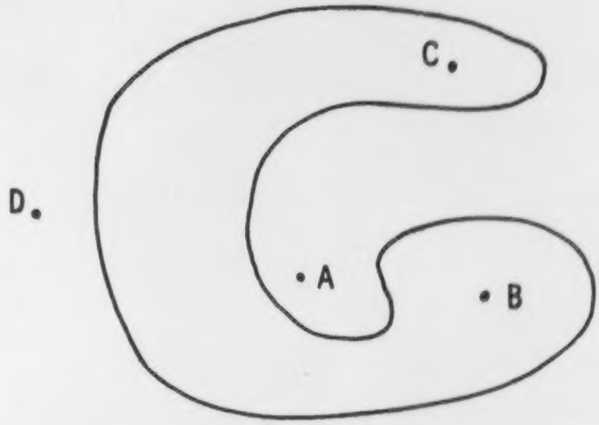
Color the triangle, but not its interior.

Color the interior using another color.

Interior and Exterior

3. Name two points in the interior of this figure. C, B

Name two points in the exterior of this figure. A, D



Without crossing the figure, can you draw a curve

from A to B? Yes [✓ No]

from A to C? Yes [✓ No]

from A to D? [✓ Yes] No

from B to C? [✓ Yes] No

from B to D? Yes [✓ No]

from C to D? Yes [✓ No]

Can any curve in a plane pass from the interior of a simple closed curve to its exterior without crossing the curve? Yes [✓ No]

Pupil's book, pages 36-37: Regions

Ideas

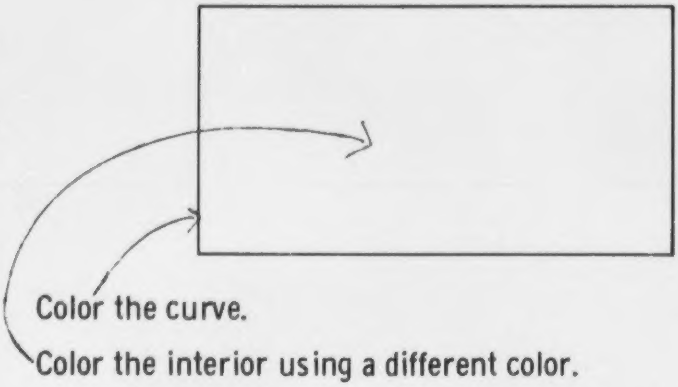
A region is the union of a simple closed curve and its interior.

Examples 1-3

Vocabulary is stressed in each example. The words should be introduced in advance as part of your demonstration technique. An expanding vocabulary chart can be used for both review and new terms. Keep children alert that the curve is an integral part of the region.

Regions

1. Here is a rectangle.



Color the curve.

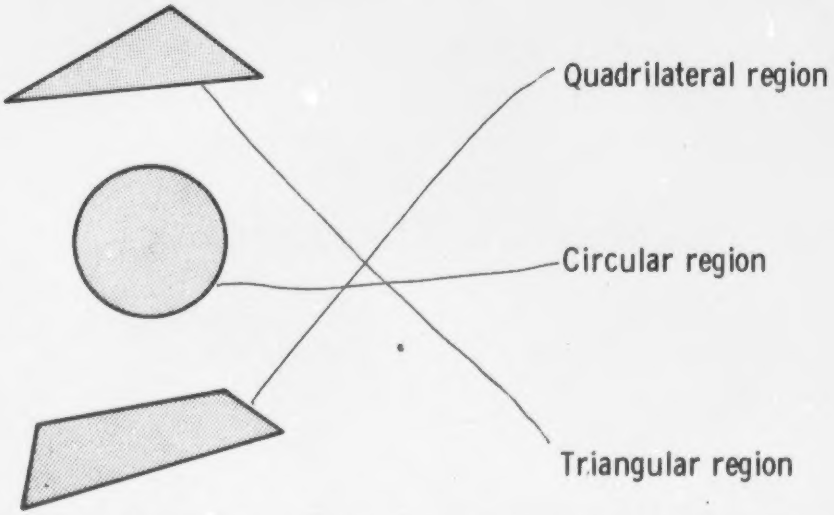
Color the interior using a different color.

When we think of a curve and its interior, we call the figure a region.

2. Below are several regions and names for regions.

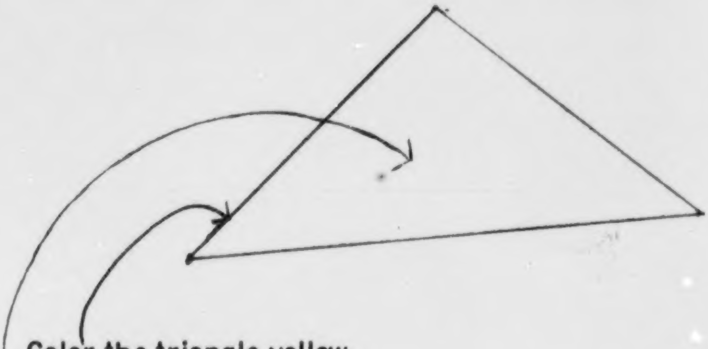
Regions will be shaded in this book.

Pair each region with its correct name.


Quadrilateral region
Circular region
Triangular region

Regions

3. Draw a triangle.

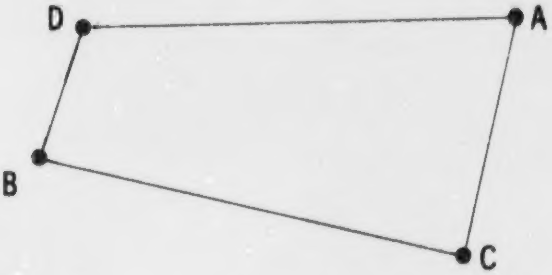


Color the triangle yellow.

Color the interior of the triangle red.

The region shown is a triangular region.

4. Draw $\overline{AD}$, $\overline{DB}$, $\overline{CB}$, and $\overline{AC}$.



Underline the correct names for the figure you drew.

(1) a simple closed curve

(2) a polygon

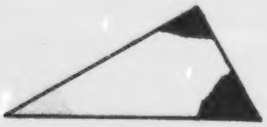
(3) a triangle

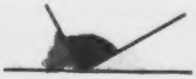
(4) a quadrilateral

(5) a quadrilateral region

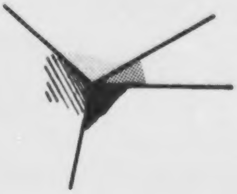
★ Enrichment Activities:

1. Have the pupils cut corner pieces from a triangular region and fit them together along a line. (Draw a triangle on construction paper. Shade corner regions before cutting to help identify them later.)

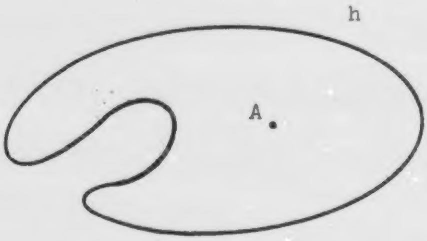




2. Similarly, verify that the corners of a quadrilateral region fit together around a point.



3. Here is a simple closed curve. It is named h. A point A is marked in its interior.



Draw another simple closed curve, m, that lies entirely in the interior of h and that contains A in its interior.

Now draw a simple closed curve, n, that lies entirely in the interior of h and entirely in the exterior of m.

I-5. Some special triangles

Objectives: To introduce isosceles, equilateral and right triangles .

Vocabulary: Right triangle, isosceles triangle, equilateral triangle .

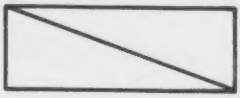
Materials: Colored chalk, paper or cardboard triangles and rectangles, pipe cleaners or sticks .

Suggested Procedure:

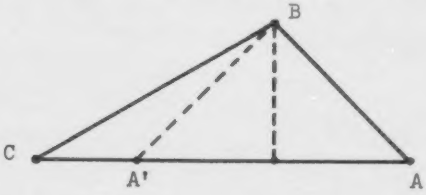
Right Triangle

When one of the angles of a triangle is a right angle, the triangle is called a right triangle. A wall, the ground, and a leaning ladder or pole will illustrate right triangles. As another example, have each child consider himself, his shadow, and the (invisible) line segment joining the two heads.

Children can cut right triangles from rectangular sheets of paper. Note that a cut through opposite corners results in two congruent right triangles or regions:



Some child may observe that the reverse is true as well: any two congruent right triangular regions can be fitted together to form a rectangular region. Another way to construct right triangles is to start with any triangular sheet and fold as shown (A onto A'):



Can a right triangle be congruent to a triangle that is not a right triangle? (No.) Can a triangle have more than one right angle? (No.)

- Isosceles triangle

A triangle with two congruent sides is called an isosceles triangle. Draw several on the board of various shapes and in various positions. Have the children make some with pipe cleaners or sticks. Discuss the apparent congruence of the angles opposite the congruent edges. Check with a paper model, tearing off one corner and matching it against the other.

Finally, display a large paper model (of an isosceles triangular region). Fold it down the middle as shown; then unfold it and mark the crease with a crayon. Note that the folding verifies the following facts.

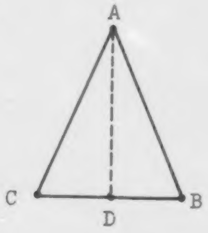
(1) Angles $\angle ABC$ and $\angle ACB$ are congruent.

(2) The angles $\angle DAB$ and $\angle DAC$ are congruent.

(3) Angles $\angle ADB$ and $\angle ADC$ are congruent and therefore are right angles. (Recall the definition of right angle.)

(4) The line segments $\overline{DB}$ and $\overline{DC}$ are congruent.

(5) The triangles $\triangle ADB$ and $\triangle ADC$ are right triangles; in fact they are congruent right triangles.



- Equilateral triangle

A triangle with all three sides congruent is called an equilateral triangle. An equilateral triangle is therefore a special kind of isosceles triangle. In an equilateral triangle, all three angles are congruent. As before, have the children make their own models.

Display a cardboard model (of an equilateral triangular region) and trace around it to draw an equilateral triangle on the chalkboard. Label the cardboard triangle A, B, C in red, and the chalkboard triangle A, B, C in yellow. Show that the triangles fit in all three positions: red A on yellow A, on yellow B, on yellow C. (If a child suggests turning the cardboard over, discuss that too.)

Can you have an isosceles triangle that is not equilateral? (Yes.) Equilateral but not isosceles? (No.) Can you have a right triangle that is not isosceles? (Yes.) That *is* isosceles? (Yes.) Is every isosceles triangle a right triangle? (No.) Can an equilateral triangle be a right triangle? (No.) Can a right triangle be equilateral? (No.)

Some important ideas are:

a. A right triangle has a right angle.

b. An isosceles triangle has two congruent sides and two congruent angles.

c. An equilateral triangle has three congruent sides and three congruent angles.

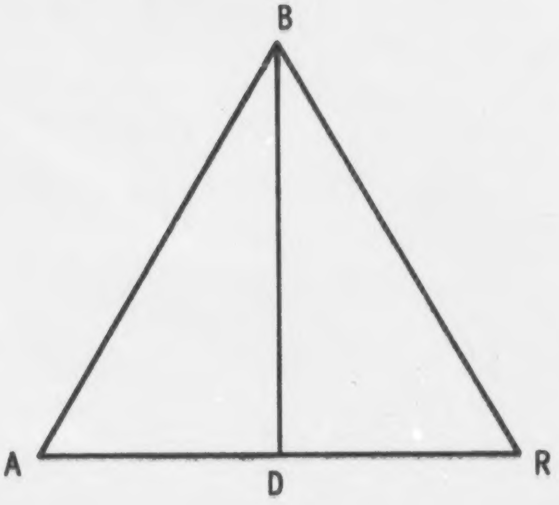
Pupil's book, page 38: Right Triangle

Ideas

A right triangle has a right angle.

Right Triangles

Here are triangle ABR and line segment $\overline{BD}$.



Are ∠BDA and ∠BDR congruent? [✓ Yes] No

Are ∠BDA and ∠BDR right angles? [✓ Yes] No

Name two right triangles. △ADB △BDR

Are these right triangles congruent? [✓ Yes] No

Pupil's book, page 39: Isosceles Triangles

Ideas

An isosceles triangle has two congruent sides.

An isosceles triangle has two congruent sides and two congruent angles.

Pupil's book, page 40: Equilateral Triangles

Ideas

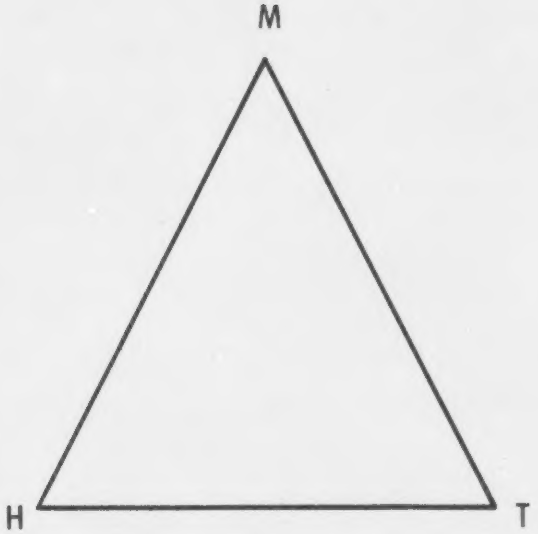
An equilateral triangle has three congruent sides.

Pupil's book, pages 41-43: Review

Ideas

The pertinent ideas shown above are reviewed.

Isosceles Triangles



1. Is $\overline{HM}$ congruent to $\overline{MT}$? [✓ Yes] No

How many congruent sides does $\triangle HMT$ have? 2

Is $\triangle HMT$ an isosceles triangle? [✓ Yes] No

2. Make a tracing of $\triangle HMT$.

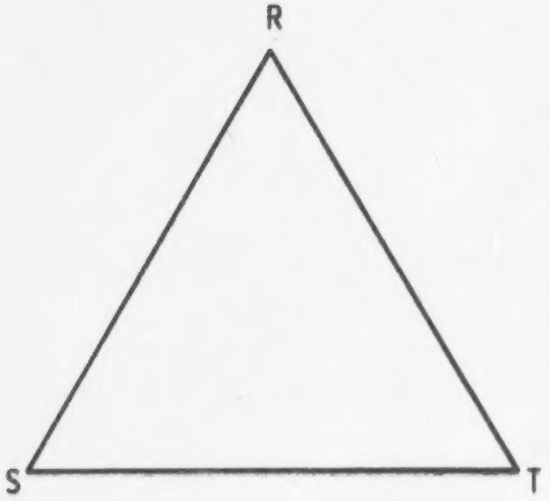
Fold it so that the tracings of $\overline{MH}$ and $\overline{MT}$ fit on each other.

Is $\angle MHT$ congruent to $\angle MTH$? [✓ Yes] No

How many congruent angles does $\triangle HMT$ have? 2

3. An isosceles triangle has 2 congruent sides and 2 congruent angles.

Equilateral Triangles



Mark off $\overline{RS}$ on the edge of a sheet of paper.

Is your copy of $\overline{RS}$ congruent to $\overline{RT}$? [✓ Yes] No

Is $\triangle RST$ an isosceles triangle? [✓ Yes] No

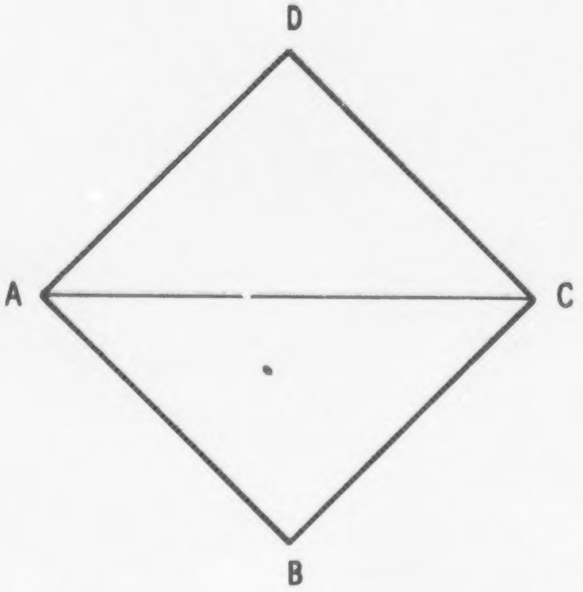
Is your copy of $\overline{RS}$ also congruent to $\overline{ST}$? [✓ Yes] No

Are the three sides of this triangle congruent? [✓ Yes] No

The special kind of isosceles triangle with all three sides congruent is called an equilateral triangle.

Is an equilateral triangle always an isosceles triangle? [✓ Yes] No

Review



1. Figure ABCD is a square.

Draw $\overline{AC}$.

Name the two triangles you see. $\triangle$ADC $\triangle$CBA

Is $\triangle$ACD an isosceles triangle? [✓ Yes] No

Name its congruent sides. $\overline{AD}$ and $\overline{DC}$

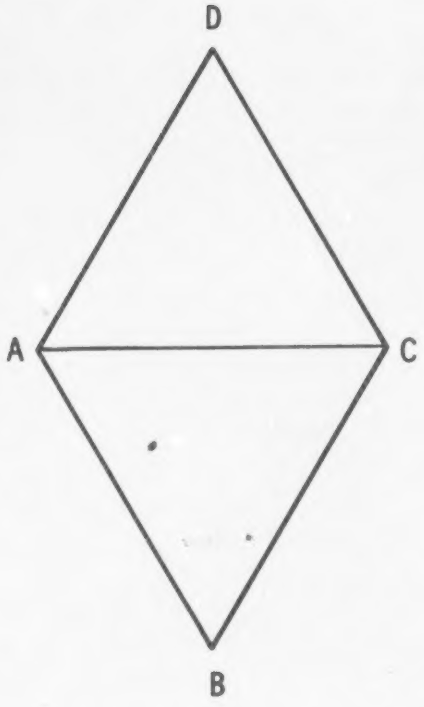
Is $\triangle$ACD a right triangle? [✓ Yes] No

Is $\triangle$ACD an isosceles right triangle? [✓ Yes] No

Is $\triangle$ACD an equilateral triangle? Yes [✓ No]

Do you think $\triangle$ACD and $\triangle$ACB are congruent? [✓ Yes] No

Review



2. Look at quadrilateral ABCD.

Draw $\overline{AC}$.

Is $\triangle ACD$ isosceles? [✓ Yes] No

Is $\triangle ACD$ a right triangle? Yes [✓ No]

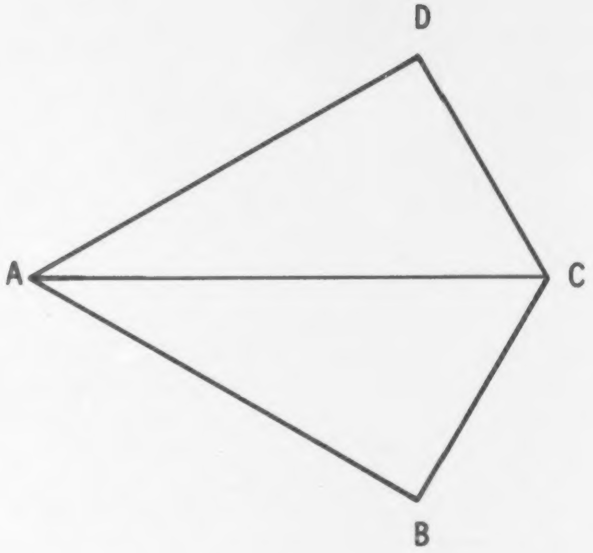
Is $\triangle ACD$ equilateral? [✓ Yes] No

Is $\triangle ACB$ equilateral? [✓ Yes] No

Are $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ congruent? [✓ Yes] No

Is ABCD a square? Yes [✓ No]

Review



3. Look at quadrilateral ABCD.

Is $\angle ADC$ a right angle? [✓ Yes] No

Is $\angle ABC$ a right angle? [✓ Yes] No

4. Draw AC above.

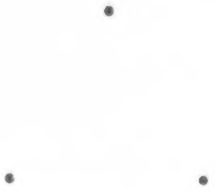
Is $\triangle ADC$ a right triangle? [✓ Yes] No

Is $\triangle ADC$ isosceles? Yes [✓ No]

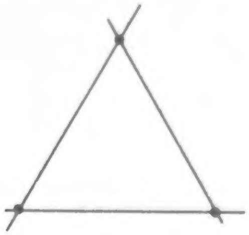
Is $\triangle ADC$ congruent to $\triangle ABC$? [✓ Yes] No

★ Enrichment Activities

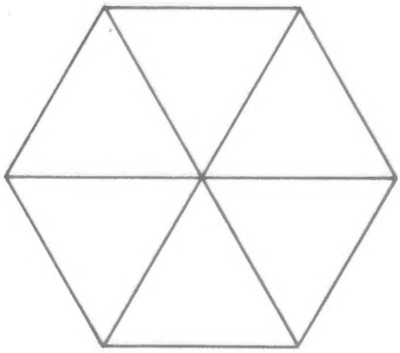
Prepare a supply of congruent equilateral triangular regions of tagboard, one for each child. The triangles should be about 4 inches on a side. Have each child make six congruent copies out of construction paper. The way to make a copy is: (1) use the tagboard model to locate the vertices on the paper; (2) connect the vertices with the help of a straightedge; (3) cut.



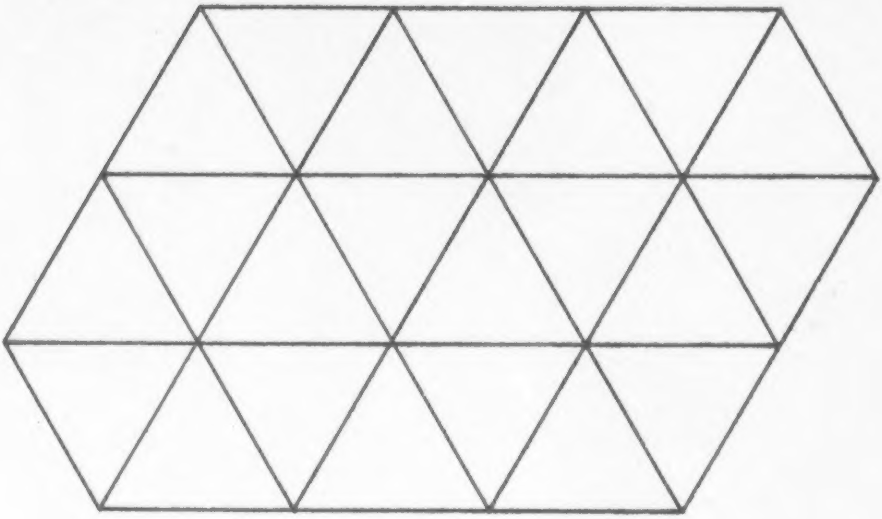
Step (1)



Step (2)



Have the children combine their equipment to construct the familiar tiling pattern:



Chapter II

ADDITION AND SUBTRACTION: REVIEW AND EXTENSION

Background

This chapter is primarily a review and extension of concepts and skills already introduced in Grades 1 and 2. In general, fuller discussions of these concepts may be found in the Background sections of the chapters for these grades. However, we give here a self-contained survey of all these ideas.

Sets. Section II-1 of the present chapter reviews ideas and terminology associated with sets. A set is simply a collection of things. The things belonging to a set are called its members. A set may be defined by some property common to its members (the set of all books on this shelf; the set of all whole numbers). Sets may also consist of quite unrelated objects and may be defined by simply listing their members, as in the set:

{the number 5, Tuesday, the moon}.

It is customary in mathematics to use braces {. . .}, as above, to symbolize the set of members listed.

It is possible for a set to have only one member. Thus, the set whose only member is the number 0 would be denoted

{0}.

There is also one special set called the empty set, which by definition has no members at all. The empty set may be symbolized simply as { }, or as $\emptyset$.

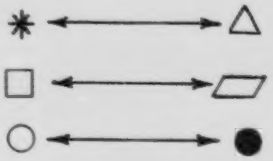
Do not confuse zero as the number of members in the empty set with the number zero as a member of the set {0}.

One set A is a *subset* of another set B if every member of A is also a member of B. Another way of saying the same thing is, "There is *no* member of A which is *not* a member of B". There are two special cases of this definition. First, *any given set is a subset of itself* (because it fits the definition of subset: every member of a given set is also a member of that set!). Second, *the empty set is a subset of every set*, because all the members of the empty set (there aren't any!) belong to that set. Or, what is the same thing, there is *no* member of the empty set which is not a member of any set.

Comparison of Sets. We can compare two sets by *pairing* the members of one set with those of the other (insofar as is possible). For example, let the two sets be

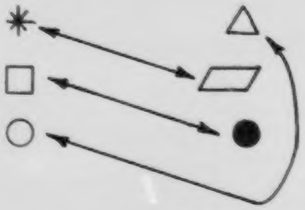
{✱, □, ○}   {   , □, ○}.

We may *pair* the members of these sets in a variety of ways. One way is:



This may be read: "The star is paired with the triangle, the square with the parallelogram, and the circle with the dot."

Another way is:

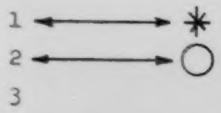


This may be read: "The star is paired with the parallelogram, the square with the dot, and the circle with the triangle".

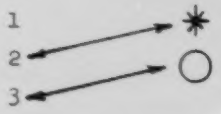
When such a pairing "comes out even", so that no member of either set is left without a partner from the other set, and also so that no member of either set has more than one partner, we say that the two sets have been placed in *one-to-one correspondence*. Note that we do not say that two sets in one-to-one correspondence are equal. We say that two sets are equal only if they are identical, i.e., only if they have *the same members*.

If two sets can be placed in one-to-one correspondence we say they are *equivalent*. Thus the two sets above, though not equal, *are* equivalent. The one common property of these two sets is their *number property*. They each have 3 members.

Let us look at another example. Let our two sets be {1, 2, 3} and { , }. There are several ways of pairing members of the first set with members of the second set in a one-to-one fashion. One of these is indicated by:



and another by:



Note that in each of these pairings, one member of the first set is not used.

When every one-to-one correspondence between two sets leaves at least one member of the first unused, we say that the first set has *more members* than the second and that the second has fewer members than the first one.

Let us exhibit the relations among these statements:

"The first set has *more members than* the second."	The *number* of members in the first set *is greater than* the number. of members in the second.
"The first set is *equivalent* to the second."	The *number* of members in the first set is *equal* to the *number* of members in the second.
"The second set has *fewer members* than the first."	The number of members in the second set is *less than* the number of members in the first.

Joining and Removing Sets. To take a set and join it to a second set means to form a new set whose members are all those of the first set together with all those of the second. The new set is called the *union* of the two given sets. Example:

First set: {a, b, c}
Second set: {p, q}
Union: {a, b, c, p, q}

In this example the first set and the second set are said to be *disjoint*: that is, these two sets have no members in common. When this happens, the number of members in the first set plus the number of members in the second is the number of members in their union; so we get (in the above example) the addition fact

$$3 + 2 = 5.$$

Thus, the joining of disjoint sets serves as an approach to the adding of numbers.

In the same way, removing from a given set one of its subsets serves as an approach to subtraction: Example:

Given set: {a, b, c, p, q}
Subset removed: {p, q}
Set remaining: {a, b, c}

Here the corresponding subtraction fact is:

$$5 - 2 = 3.$$

When we first join, and then remove from the union, the same set (as we did in the above two examples), the result is the set with which we started. This inverse relationship between first joining and then removing a given set is reflected in the inverse relationship between first adding and then subtracting a given number. The above examples indicate that when we take 3 and add to it 2, and then subtract 2 from the sum, we get back the same number, 3.

We need to look at this in the other sense, i.e., starting with a given set, then removing some subset, and finally replacing the same subset. This will suggest a connection with first subtracting a number from a given number to obtain a difference, then adding the same number to the difference to get the original given number as the final result. Let us start with the given set {a, b, c, p, q}. It has five members.

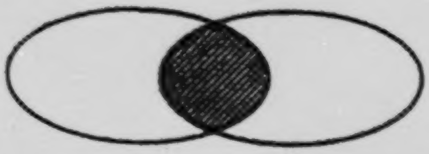
If we remove the subset {p, q} with its two members, the elements not removed form the set {a, b, c} with its three members. We are reminded of the subtraction fact:

$$5 - 2 = 3.$$

If now we join {p, q} to {a, b, c}, we get the set {a, b, c, p, q} as the union. So we have the original given set as the union. Now we are reminded of the addition fact:

$$3 + 2 = 5.$$

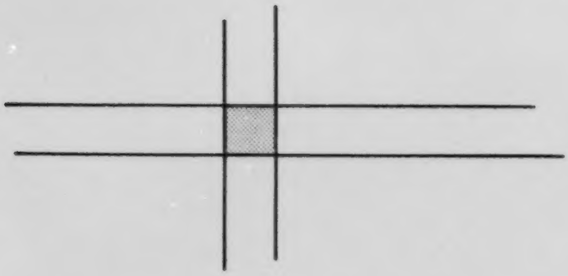
Intersections of Sets. Section II-1 introduces the idea of intersection of sets. The intersection of two sets is the set consisting of the members which belong to both sets as suggested below:



More precisely, the intersection of one set with another set has as its members those things which belong both to the first set and to the second. Example:

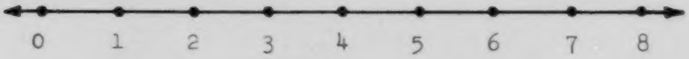
First set:	{a, b, c, d}
Second set:	{c, d, e}
Intersection:	{c, d}

Since the common part of two streets is called their "intersection", this fact may help children grasp the idea and the term.



There is one subtle point to note in connection with intersections: the set of all members common to two disjoint sets is empty, so the intersection of two disjoint sets is the empty set. Thus, we may speak of the intersection of any two sets. It may or may not be empty.

The Number Line. We may represent the whole numbers 0, 1, 2, 3, ... by means of equally spaced points on a line, as shown below:

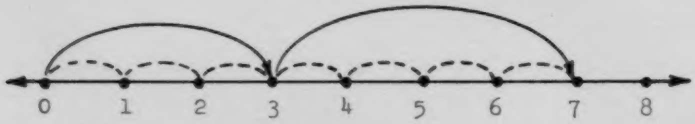


The number line is useful in many ways. For one thing, it helps visualize the comparison of numbers: "greater than" corresponds to "to the right of", and "less than" corresponds to "to the left of".

Addition is easily visualized on the number line. The result,

$$3 + 4,$$

of "taking 3 and adding to it 4" is pictured on the number line below as taking 3 steps to the right (beginning at 0) and then taking 4 more steps to the right, thus:



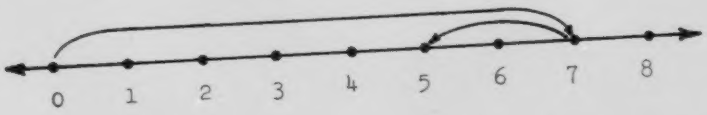
The fact that in this way we arrive at the point labeled "7" illustrates the addition fact:

$$3 + 4 = 7.$$

The subtraction fact:

$$7 - 2 = 5$$

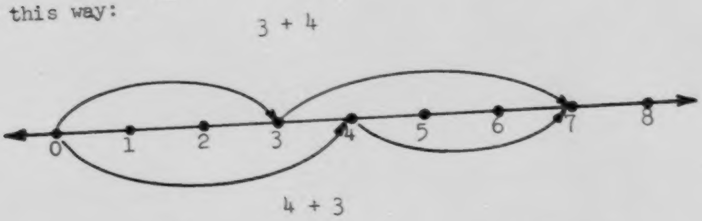
can be illustrated thus:



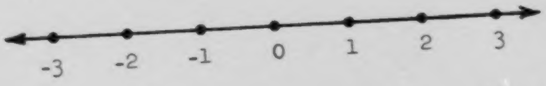
The number line also can be used to show the commutative property of addition. For example, we can visualize the statement

$$3 + 4 = 4 + 3$$

in this way:



Later the number line will offer a very natural way of picturing the extension of the number system to include the negative integers,



the rational numbers, and, at a considerably later stage, the irrational numbers (like $\sqrt{2}$ and π).

The Addition Chart. An objective of the present chapter is mastery of the basic addition facts. These facts have already been introduced in Grade 2. In Section II-3 of the present chapter they are summarized in an addition chart or table. In using this chart, the first number of the pair indicates the row, and the second the column, in which the sum of these numbers will be found. For instance, to read the fact

$$5 + 8 = 13$$

from the table, we look at the intersection of row 5 and column 8 to find the sum 13.

If we interchange row and column (that is, look in row 8 and column 5, instead of in row 5 and column 8), we again find the same sum 13:

$$8 + 5 = 5 + 8,$$

which is an instance of the commutative property of addition. In general, the commutative property of addition is reflected on the chart by noting that interchanging row and column does not change the number found in the table.

II-1. Sets: joining and removing

Objectives:
(a) To review important ideas associated with sets and subsets.
(b) To extend the ideas of joining and removing sets, the relation of addition and subtraction to these set operations, and the use of equations to describe these relationships.
(c) To introduce the intersection of sets.

Vocabulary: (Review) braces, intersection, disjoint, set, member, empty set, join, union, subset, removal, equation.

Materials: The materials below are listed a, b, and c to correspond to specific suggested procedures in this section.
(a) A variety of manipulative objects including materials for the flannel board (e.g., animals, trees, fruits, stars, flags, and hearts); blocks, disks, books of various sizes and colors, etc.
(b) (None.)
(c) 9 colored autos or pieces of paper: 4 red, 2 blue, and 3 green with masking tape on the back.
A red crayon, a blue crayon, and a green crayon (children).

Suggested Procedure:

(a) Although the vocabulary is not new and the ideas will be extended in subsequent sections of this chapter, an extensive review is suggested.

Assemble various kinds of sets in order to provide opportunities for precise descriptions; for example, "a set of blue stars on our flannel board" instead of "a set of stars". Using three sets of story books, two of which have different colored covers and the third set with only blue covers, you could ask the children to choose a set of story books and then to choose the set of story books with blue covers. In the first case they might choose any set and be right, but the more specific description allows for the choice of only one set.

- Select a set of three or four boys and girls. Write their names on the chalkboard; for example: Mary, Jane, Bob, Bill. Ask children (other than the members of the set) if they belong to this set. Emphasize the idea that only Mary, Jane, Bob, and Bill belong to the set and therefore are called members of the set. It should be clear that the set consists of Mary, Jane, Bob, and Bill simply because this was the set selected by their teacher for discussion. Children should not believe that members of a set must be related in some particular way other than the fact that they are members of the set.

  Follow by considering all the children in the classroom. Ask the children to describe that set and name people or objects that do not belong or are not members of the set. Then consider possible subsets of the set of children.

  > What name do we use for the set of girls in the classroom who belong to the set of children in our classroom? (Subset.)
  >
  > Describe other subsets. (Set of boys in our classroom, set of children with blue eyes, set of children with brown hair, etc.) Can

you call the set of children in the first row a subset of the set of children in our classroom? (Yes.) How do you know? (Because every child in the first row belongs to the set of children in our classroom.)

Is the set of teachers in this room a subset of the set of children? (No.) Determine if other people and objects are subsets of the set of children. (e.g., books, chalk, parents, principal, etc.)

- Continue the discussion using the Pet Shop picture on page 44 of the pupil's book. Have children turn to this picture and describe things they see in the picture. Encourage the use of "set language" such as, "a set of 3 puppies," etc.

Pupils who have visited pet shops may be given an opportunity to tell some of the things they have seen there. Also provide an opportunity for children to tell about some of their own pets--particularly unusual pets they may have.

Redirect children's attention to the picture in the pupil textbook. Use questions such as the following ones to inventory children's grasp of important ideas associated with sets. Be watchful for ideas that may seem to need clarification for some children.

Describe the set in the picture. (Set of animals in the window of a pet shop.)
Is the set of puppies a subset of this set? (Yes.)
How do you know? (Each puppy belongs to the set of animals in the pet shop window.)
How many members in the subset of puppies? (3.)
Can you describe a subset of four members? (Cats.)
Of two members? (Monkeys.) of a single member? (Parrot.)

Describe the set of turtles in the Pet Shop window. (The empty set.)

How many members are there in the set of all pets in the window? (10.)

Describe the set of fish in the Pet Shop window. (The empty set.)

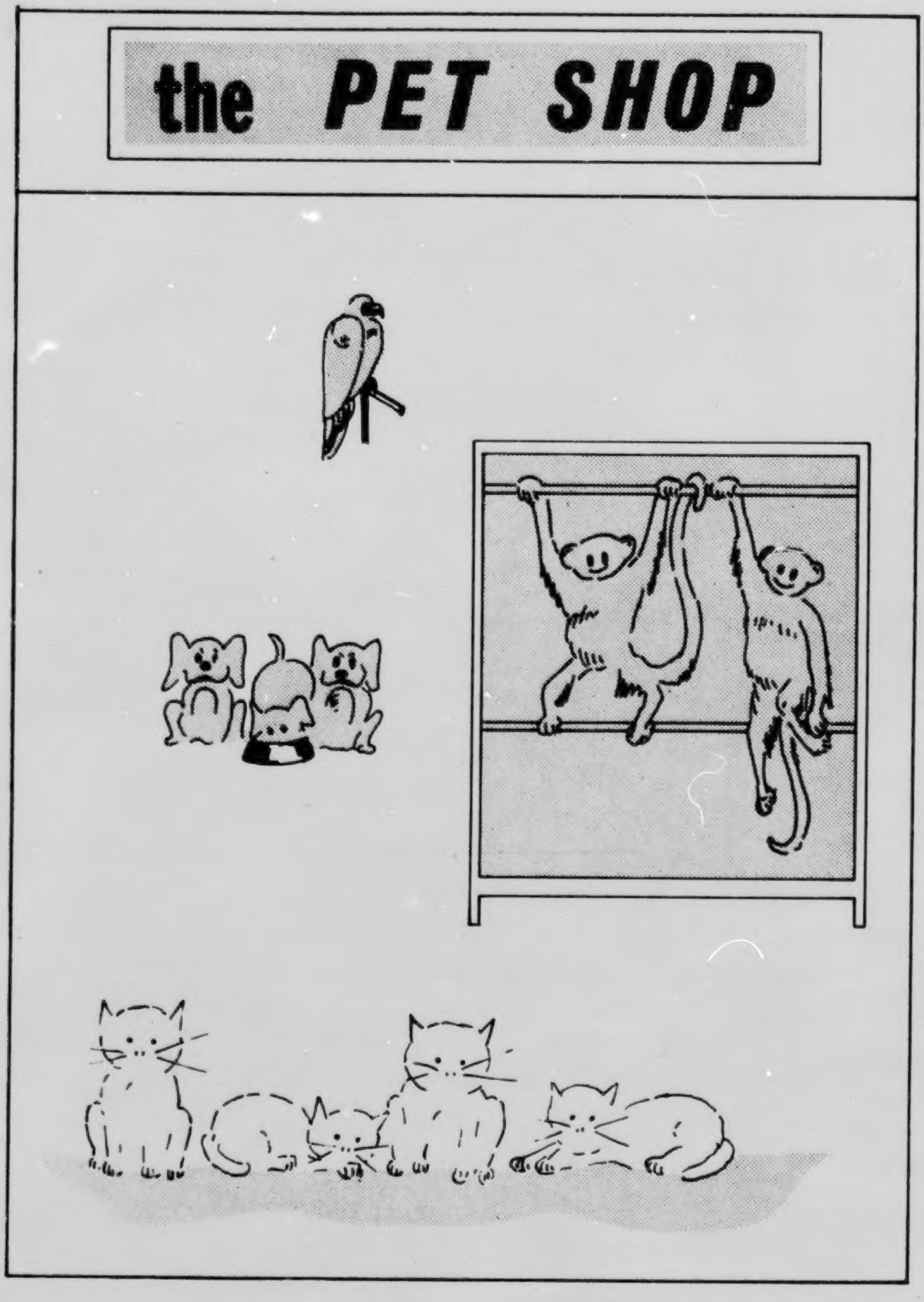
How many members in the set of pets on the floor? (7.)

How many members in the set of pets not on the floor? (3.)

Pupil's book, pages 44 and 45

Page 44 is to be used in the discussion above.

Page 45 may be used by children independently.


the PET SHOP

Subsets

GIRL	APPLE	[✓ GREEN]	BANANA
BLUE	[✓ GRAPE]	BABY	[✓ PURPLE]
[✓ GRASS]	BOOK	[✓ PLATE]	BLACK

Set A

1. The words in the box that begin with a are a subset of Set A.
List the members of this subset. Apple

2. The words that begin with b are also a subset of Set A.
List the members of this subset. Banana, Blue,
Baby, Black, Book

3. Ring the words that begin with p. How many members in this
subset? 2

4. Describe the subset whose members are words that begin with z.
the empty set

5. Ring the words that begin with gr. How many members in this
subset? 3
List the members. Green, Grape, Grass

(b)

Union of Sets

Illustrate the union of sets using manipulation material (e.g., the set of chalk joined with the set of erasers. Following by using the set of boys joined with the set of girls.) Describe the new set, the union of these two sets, as the set of children in our room.

Gradually move away from manipulation materials and the children themselves to sets which can be represented on the chalkboard.

Set A	Set B	Union of A and B
× △ *	▭ ○ ●	× △ * ▭ ○ ●

Continue with other illustrations such as:

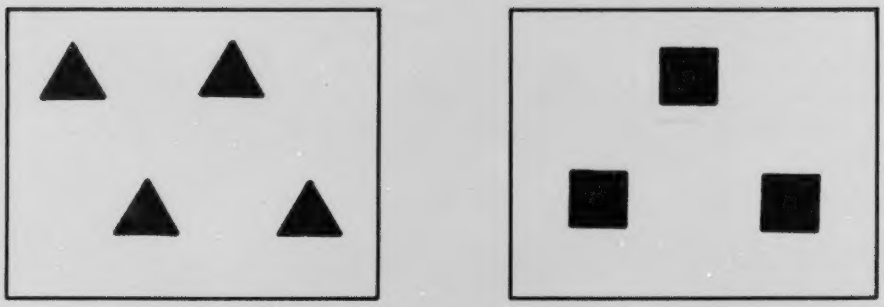
Set A	Set B	Union of A and B
a, b, c	d, e, f	a, b, c, d, e, f

In every case, ask how many members in Set A, in Set B, and in the union of A and B. This will lead naturally into the writing of an equation which describes the set situation. In this case $3 + 3 = 6$.

Pupil's book, pages 46 - 47:

> Use these pages as a check on the children's grasp of ideas related to the union of sets and the writing of an equation that describes the union of two sets.

Union of Sets


Set A   Set B

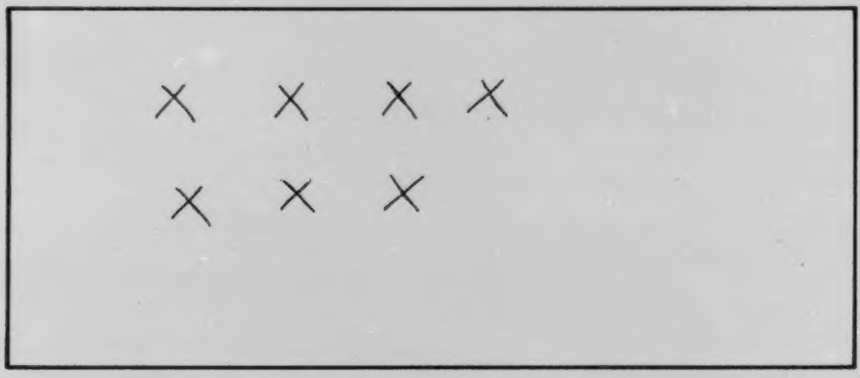
1. How many members are in Set A? 4

2. How many members are in Set B? 3

3. Think of joining Set A and Set B.

   When we join two sets we have a new set called the union of the two sets.

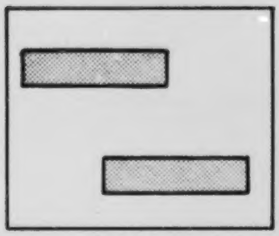
   Draw a picture for the union of sets A and B.



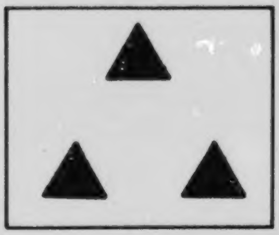
4. How many members are in the union of sets A and B? 7

5. Write an equation for the two sets and their union. 4 + 3 = 7

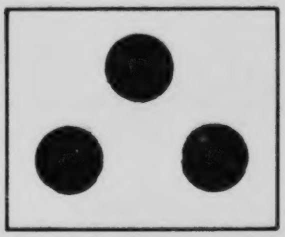
Union of Sets



Set X



Set Y



Set Z

6. How many members are in Set X? 2

7. How many members are in Set Y? 3

8. Draw a picture for the union of sets X and Y.

X X X X X

9. Write an equation for X and Y and their union.

2 + 3 = 5

10. How many members are in Set Z? 3

11. Draw a picture for the union of sets Y and Z.

X X X
X X X

12. Write an equation for Y and Z and their union.

3 + 3 = 6

- Removing a subset

Place a set of colored blocks on the desk - 1 green, 2 blue, 3 yellow, and 4 red blocks (or some similar arrangement). Have the set described and the number of members in the set identified.

Ask the children if each set of colored blocks is a subset of the set of blocks. Then ask one child to remove from the set of blocks the subset of yellow blocks and tell how many blocks are in this subset.

Ask another child to describe the set that is left. Continue the discussion so that the children see that from the set of green, blue, red and yellow blocks (10 members) the subset of yellow blocks (3 members) was removed; it was not removed from the set of green, blue, and red blocks (7 members).

One way to help children see what has happened is for you to remove the subset of yellow blocks from the table so that children see only green, blue, and red blocks. Ask a child to remove a subset of yellow blocks from this set. (Of course, he cannot do this.)

Returning the whole set to the desk, ask a child to remove the subset of blue blocks and describe the set that is left.

> Mary, will you describe this action? What took place? What did John do?

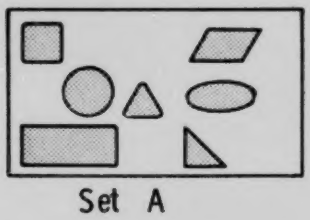
Be sure that the child responds that from the set of green, blue, yellow and red blocks, John removed the subset of blue blocks. Follow this with a description of the action in terms of the number of members involved. (Started with a set of 10 members. Removed a subset of 3 members. The set remaining has 7 members.)

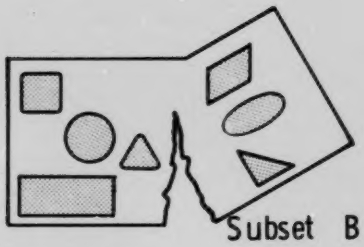
Follow a similar procedure with the subset of green blocks, the subset of red blocks and the empty set.

Suggest that a record of what has been done be completed on the chalkboard.

Removing a Subset

1. Look at these pictures.


Set A


Subset B

How many members are in Set A? 7

2. How many members are in the subset being removed? 3

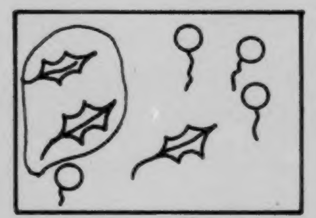
3. Draw a picture of the set that would be left when Subset B is removed from Set A.

X X X X

4. How many members are in the set remaining when Subset B is removed from Set A? 4

5. Write an equation which describes the set remaining. 7-3 = 4

6. Look at these pictures. Ring a subset in each picture and write an equation for the set remaining.


7-2 = 5

A B C
D E F
G H I

9-3 = 6

Jane Bob
Sally Mary
Joe Ann
Bill Charles

8-4 = 4

Number of members in the set	Number of members in the subset removed	Number of members in the set remaining
10	3 (yellow)	7
10	2 (blue)	8
10	1 (green)	9
10	4 (red)	6
10	0 (empty set)	10

Ask different children to write the equation for each row on the chart. The notation should look like this:

$$10 - 3 = 7$$
$$10 - 2 = 8$$
$$10 - 1 = 9$$
$$10 - 4 = 6$$
$$10 - 0 = 10$$

Emphasize the fact that the union of the subset removed and the set remaining is the original set. Equations describing these set situations can be given by the children, i.e., $3 + 7 = 10$, $2 + 8 = 10$, etc.

Pupil's book, page 48: May be used at this time.

More about Joining and Removing

Draw on the chalkboard a set of 7 simple objects of your own choice. Ask children to describe the set. Draw another set of 7 different objects and ask the children to describe this set.

Mention that it would be convenient to be able to refer to these sets in a simple way, without describing them. Ask for suggestions. Hopefully, some child will suggest "Set A" and "Set B" as used in previous lessons.

Label the sets drawn on the chalkboard "Set A" and "Set B". Indicate to the children that we often use capital letters, in this instance A and B, to refer to particular sets.

Display a set of things from your desk, such as a mathematics book, a pencil, a chalk box and a ruler. Write the following on the chalkboard:

Set D = {mathematics book, pencil, chalk box, ruler}

Explain that you have used "braces" to enclose the names of the members of the set of things from your desk.

Call pupils' attention to the fact that since we referred to the first set as "Set A" and the second set as "Set B", we used some other letter to refer to the set of things from your desk. If we used the same name for different sets, we could not distinguish between them easily.

Now write the following on the chalkboard:

{pencil, ruler, mathematics book, chalk box}

Bring out the idea that what you have written on the chalkboard was intended to name the members of the same set of things from your desk, but in a different order than before. Therefore, it is the same set as "Set D" and not a different set. The order in which we list the members of a set is immaterial.

Pupil's book, pages 49 - 51: May be used at this time. Your own judgment will guide you in deciding to what extent children can work independently, and to what extent they will need to work together with you.

Further Activities

1. More able children may find it interesting to form all the subsets of a given set. Have them consider a set of children's names, for example:

   {Bob, Tom, Sue}

   Here are listed the 8 possible subsets that can be formed from the given set:

   Subsets with 1 member each: {Bob}
   {Tom}
   {Sue}

Subsets with 2 members each: {Bob, Tom}
{Bob, Sue}
{Tom, Sue}
Subsets with 3 members each: {Bob, Tom, Sue}
Subsets with 0 members each: The empty set
(Don't forget this one!)

For this set of 3 things--and for any set of 3 things--there are 8 subsets.

2. The following table summarizes the number of subsets for sets having a specified number of members:

Number of members in the set	Number of possible subsets
0	1
1	2
2	4
3	8
4	16
5	32
etc.	etc.

(Using more sophisticated mathematical symbolism: if n represents the number of members in a set, 2^n represents the number of possible subsets of that set.)

Union of Sets

1. Set A = {lamb, pig, dog}

   Set B = {cow, cat}

   Ring the set that is the union of sets A and B.

   {lamb, horse, pig, dog, cat}   {lamb, pig, dog, cat, goat}

   [✓ {lamb, pig, dog, cat, cow}]   {cow, cat, lamb, dog, fish}

2. Set C = {book, pencil, eraser, crayon}

   Set D = {clip, tape, ruler}

   Ring the set that is the union of sets C and D.

   {crayon, ruler, pencil, eraser, tape}

   [✓ {clip, ruler, book, crayon, pencil, eraser, tape}]

   {tape, ruler, book, pencil, eraser, crayon, chalk}

3. Set E = {rubber, tin, doll}

   Set F = {ball, kite, bat, car}

   Set G is the union of sets E and F.

   Ring set G.

   {rubber, tin, ball, car, doll, car, kite}

   {car, rubber, tin, doll, ball, kite, cap}

   [✓ {kite, doll, ball, rubber, tin, car, bat}]

Removing a Subset

1. Set R = {dress, hat, sock, shoe, coat}
Set T is a subset of Set R.
Set T = {shoe, sock}
Ring the set remaining when Set T is removed from Set R.
{sock, shoe} [✓ {coat, hat, dress}] {hat, shoe, coat}

2. Set V = {doll, wagon, ball, house, crayon}
Set W = {ball, crayon}
Ring the set remaining when Set W is removed from Set V.
{house, dog, cat, ball} {crayon, ball} [✓ {wagon, doll, house}]

3. Set F = {0, 1, 2, 3, 4, 5, 6}
Set G = {6, 4, 2, 0}
Ring the set remaining when Set G is removed from Set F.
{3} {2, 3, 4, 7} [✓ {5, 1, 3}]

4. Set H = {0, 1, 2, 3, 4, 5, 6, 7, 8}
Set K is the set of numbers less than 5.
List the members of Set K. 0, 1, 2, 3, 4
Ring the set remaining when Set K is removed from Set H.
[✓ {5, 6, 7, 8}] {6, 7, 8} {0, 1, 2, 3, 4}

5. Set P = {11, 12, 13, 14, 15}

Set X is the set of numbers less than 12 in set P.

List the members of Set X. 11

Ring the set remaining when Set X is removed from Set P.

[✓ {12, 13, 14, 15}] {13, 14, 15} {14, 15}

6. Set M = {20, 21, 22, 23, 24, 25}

Set H is the set of numbers greater than 23 in set M.

List the members of Set H. 24, 25

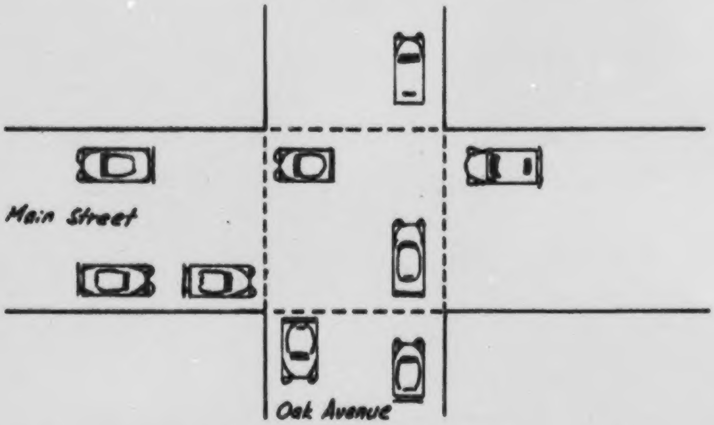
Ring the set remaining when Set H is removed from Set M.

[✓ {20, 21, 22, 23}] {24, 25} {20, 21, 22}

- (c) Intersection of sets

Thus far the experiences with joining sets have been restricted to the case of disjoint sets; that is, to sets having no members in common. Now we are going to consider sets which may or may not be disjoint.

To present the idea of the intersection of two sets write the word intersection on the board and ask if anyone can pronounce the word, tell what it means, etc. If so, in what context? The expected response is that which relates to two streets. If it is not given, lead them into a discussion of the intersection formed by two streets. For example, one street might be called Main Street and the other Oak Avenue. Draw this diagram on the board, and place the colored autos as indicated:



Identify on the diagram: Main Street, Oak Avenue, and the dashed region called the intersection of Main Street and Oak Avenue.

Now ask children to identify the members of the set of cars that are on Main Street. (The 2 blue cars are members of this set as well as the 4 red cars.) (6.)

Next ask children to identify the members of the set of cars that are on Oak Avenue. (The 2 blue cars are members of this set as well as the 3 green cars.) (5.)

Which cars are members of both sets? (2 blue cars.)
Which cars are in the intersection of Main Street and Oak Avenue? (2 blue cars.)
The set, each of whose members belongs to both of two sets, is called the intersection of these two sets.

- Develop another illustration of intersection of sets by using two sets of children having several members in common. If at all possible, use overlapping sets drawn from the classroom environment, such as a reading group and a mathematics group. Record the members of the sets on the chalkboard, such as:

Set A: {Tom, Joe, Sue, Bill, Jane}
Set B: {Bob, Sue, Ellen, Jane, Ted, Tom}

Which members of Set A are also members of Set B? (Tom, Sue, Jane.)
Which members of Set B are also members of Set A? (Tom, Sue, Jane.)

Name the members of the set that is the intersection of Set A and Set B. (Tom, Sue, Jane.)

How many members are in Set A? (5.)
How many members are in Set B? (6.)
How many members are in the union of Sets A and B? (8.)

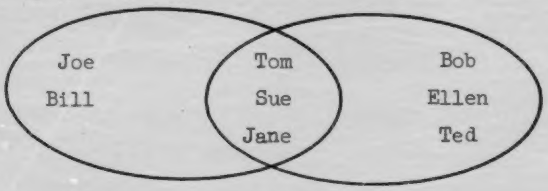
If the children say 11, which will probably happen, ask the children who are members of Set A and B to stand. They will discover that only 8 children are standing. If they cannot explain that the difficulty arises because Sue, Tom, and Jane are members of both sets, you may help them in this way:

Represent 2 sets on the chalkboard:

x x x	x x x
x x	x x x
Set A	Set B

When two sets have no members in common, we can find the number of members in the union of these sets by adding the number of members in the first set and the number of members in the second set. In this case $5 + 6 = 11$. There are 11 members in the union of Sets A and B.

The situation in which we used children is different. We can say that the sets look like this:



These sets are intersecting sets. That means that they share some members. They are not disjoint sets. When sets have no members in common, we call them disjoint sets.

Bring out the idea that we can add the number of members in two sets to find the number of members in the union, only if the two sets are disjoint. Ask the children if the two sets in the second illustration are disjoint. (No.) Then ask how we can find the number of members in the union of two sets that are not disjoint. Emphasize that we can add the number of members in the two sets and then subtract the number of members in the intersection.

Write: $(5 + 6) - 3 = 8$.

- As a final illustration use a situation in which two sets are disjoint; i.e., they have no members in common. For example:

Set X: {desk, pencil, button}
Set Y: {chair, pen, hat, ball}

Which members of Set X are also members of Set Y? (None.)

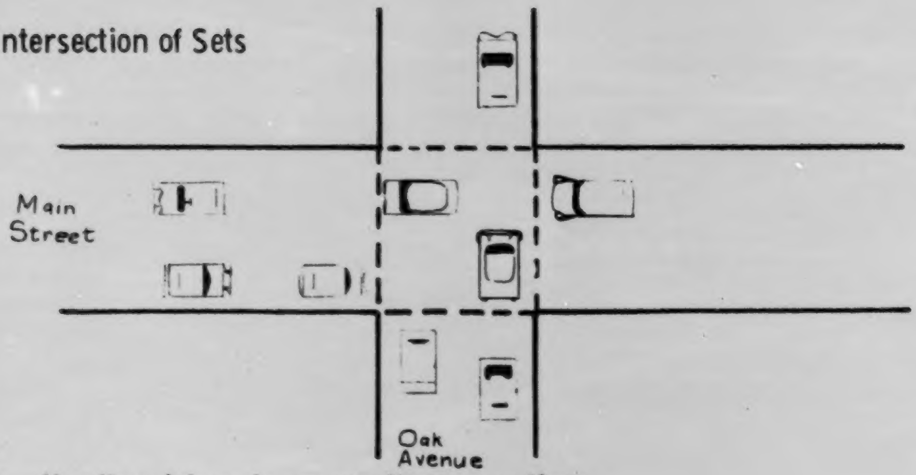
Which members of Set Y are also members of Set X? (None.)

> Name the members of the set that is the intersection of Set X and Set Y. (No members.) Are Set X and Set Y disjoint sets? (Yes.) What is the intersection of two disjoint sets? (The empty set.)

Redirect children's attention to the Main Street--Oak Avenue diagram on the chalkboard. Ask them to open their textbooks to page 52. Work together in answering the questions about the picture.

Pupil's book, pages 52 - 54: May be used for either guided or independent work, depending upon the ability of the children in your class.

Intersection of Sets



1. Use the picture to answer these questions.

How many cars are on Main Street but not on Oak Avenue? 4
Color each of these cars red.

How many cars are on Oak Avenue but not on Main Street? 3
Color each of these cars green.

How many cars are on Main Street and on Oak Avenue at the same time? 2
Color each of these cars blue.

Total number of cars in the picture: 9

Total number of cars on Main Street: 6

Total number of cars on Oak Avenue: 5

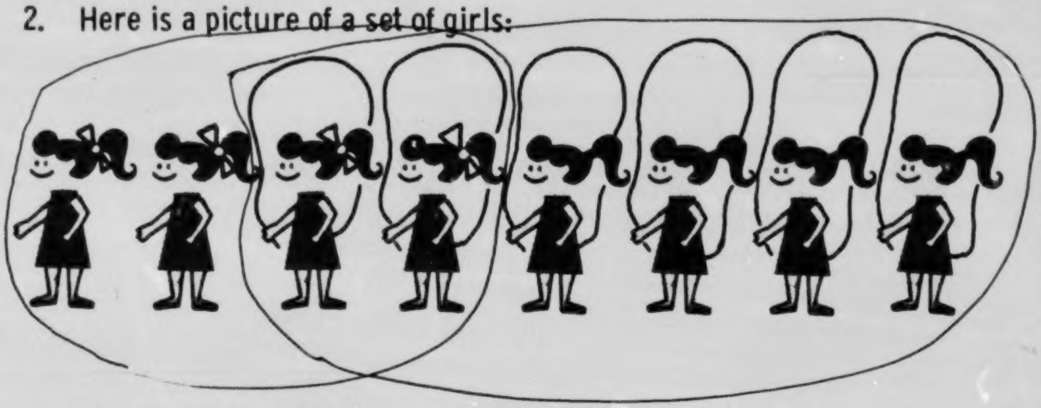
Number of cars in the intersection of Main Street and Oak Avenue: 2

Explain each of these sentences in relation to the picture:

4 + 3 + 2 = 9

6 + 5 = 11, and 11 - 2 = 9.

2. Here is a picture of a set of girls:



How many girls are in each of these subsets:

The set of girls with bows and jump-ropes. 2

The set of girls with bows but without jump-ropes. 2

The set of girls with jump-ropes but without bows. 4

Are each two of these three sets disjoint? Yes

Write an equation for the number of girls all together using the numbers of girls in the three subsets:

2 + 2 + 4 = 8

Draw a ring around each of these sets:

The set of girls with bows. This set has 4 members.

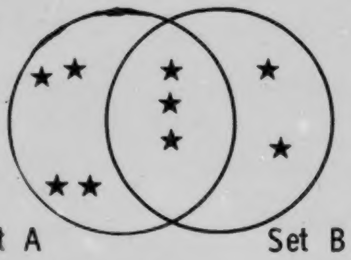
The set of girls with jump-ropes. This set has 6 members.

Are these two sets disjoint? No

How many members are in the intersection of these two sets? 2

Write an equation for the picture. (4 + 6) - 2 = 8

3. Here are some sets:



Are sets A and B disjoint sets? No

How can you tell? They have members in common.

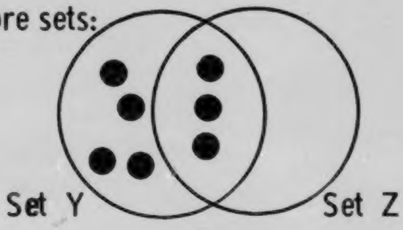
How many members are there in set A? 7

How many members are there in set B? 5

How many members are there in the intersection of set A and set B? 3

How many members are there in the union of set A and set B? 9

4. Here are some more sets:



How many members in Set Y? 7

How many members in the intersection of sets Y and Z? 3

There are 12 members in the union of sets Y and Z.

How many members in set Z? 8

II-2. Comparison of sets; order among numbers

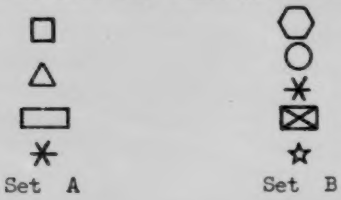
Objective: To review comparison of sets and order among numbers.

Vocabulary: Fewer than, equivalent, as many as, more than, less than, equal to, greater than; and the symbols >, =, <.

Materials: Number line.

Suggested Procedure:

Represent two sets on the chalkboard such as:



> If you are not sure which set has more members, how can you find out, if counting is not permitted? (Pair the members.)
> Mary, will you do that for us?

When the child has finished, ask how the sets can be described without using numbers. (The set on the left has fewer members than the set on the right and the set on the right has more members than the set on the left.) Bring out the fact that the pairing shows this since after pairing, one member of the Set B was not paired with a member of Set A.)

> How many members are in Set A? (4.)
> How many members are in Set B? (5.)
> May we say then that 5 is greater than 4? (Yes.)
> May we say that 4 is less than 5? (Yes.)

Write on the chalkboard:

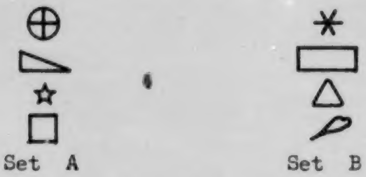
4 5

Ask if any child remembers the symbol we use to indicate that 4 is less than 5. If someone does, have him write the symbol between the 4 and the 5. If not, then indicate we use the symbol $<$. Follow the same procedure with the symbol for greater than.

Ask the children to read both statements: $4 < 5$ and $5 > 4$.

> (Four is less than five.)
> (Five is greater than four.)

Represent two other sets on the chalkboard, such as:



Ask a child to pair the members of the two sets.

> Does Set A have more members than Set B? (No.)
> Does Set A have fewer members than Set B? (No.)
> How can you describe the relationship? (Set A has as many members as Set B.)
> There is a word that we use to describe this situation. Does anyone remember what it is? (Equivalent.)

If any child says that the two sets are equal, a brief review of the relation of equality of sets should be given.

> We say that $7 + 2 = 9$. What does equal mean? (Another name for the same thing. "7 + 2" is another name for "9".)
> Is "Set B" another name for "Set A"? (No.)
> How do you know? (There are different things in the sets.)

When sets have the same number of members, we say they are equivalent. They must have exactly the same members if they are equal. These sets are equivalent, but not equal. If necessary, select two sets of children in the classroom.

{John, Joe, Bill} {Mary, Ann, Jane}

Point out that these sets are equivalent since they have the same number of members. They are not equal, however, since the set of Mary, Ann, and Jane is not another name for the set of John, Joe, and Bill.

Return to the equivalent sets represented on the chalkboard. Ask how many members are in Set A. Write 4 on the chalkboard. Then ask how many members are in Set B. Write 4 on the chalkboard. Follow by asking a child to write a symbol between the numerals to indicate the relation between the numbers.

$4 = 4$

Now write several pairs of numerals on the chalkboard, such as

6	8
24	24
32	55
61	43
40	41
17	57
2	18
65	12
99	99

Ask different children to write the symbol (<, =, >) which indicates the relation between each pair of numbers. When a child writes the symbol, he should read the sentence. For example, "Six is less than eight." "Sixty-one is greater than forty-three," and so on.

- Refer to a picture of the number line to use the concepts of greater than and less than at the same time.

> Can anyone tell me from the way the number line is drawn which number is greater, 7 or 9 ? (Nine. It is to the right of 7.)

Here is 3 on our number line. In which direction should we move if we wish to find 8 ?
(To the right.)
What whole numbers will we pass before we get to 8 ? (4, 5, 6, 7.)
What are the whole numbers between 3 and 8 ?
(4, 5, 6, 7.)
Are these numbers greater than 3 ? (Yes.)
Are these numbers less than 8 ? (Yes.)
Then, we can say that 4, 5, 6, and 7 are greater than 3 and less than 8.

Point to the number line. Remind the children that we are thinking only of whole numbers.

What numbers are greater than 2 but less than 10 ? (3, 4, 5, 6, 7, 8, 9.)
Let us see if we can answer this question without using the number line. What numbers are greater than 11 and less than 15 ?
(12, 13, 14.)

Follow with several other exercises from the following:

What numbers are greater than 5 but less than 7 ?
What numbers are greater than 8 but less than 15 ?
What numbers are greater than 17 but less than 27 ?
What numbers are greater than 29 but less than 35 ?
What numbers are greater than 45 but less than 54 ?
What numbers are greater than 10 but less than 11 ?

Finish with a question such as, "What numbers are greater than 12 but less than 4?"

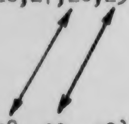
Responses to this last question should provide opportunity for further discussion.

Pupil's book, pages 55 - 58: May be used for either guided or independent work, depending upon the ability of the children in your class. For page 57 "Using the Number Line", pupils should write above each "number line" the necessary numerals.

Page 58 is concerned with the intersections of sets of numbers. It might be well to discuss a similar exercise with the children before they begin work on the page. For example,

Set A is the set of whole numbers greater than 25 but less than 30.

Set A = {26,27,28,29}



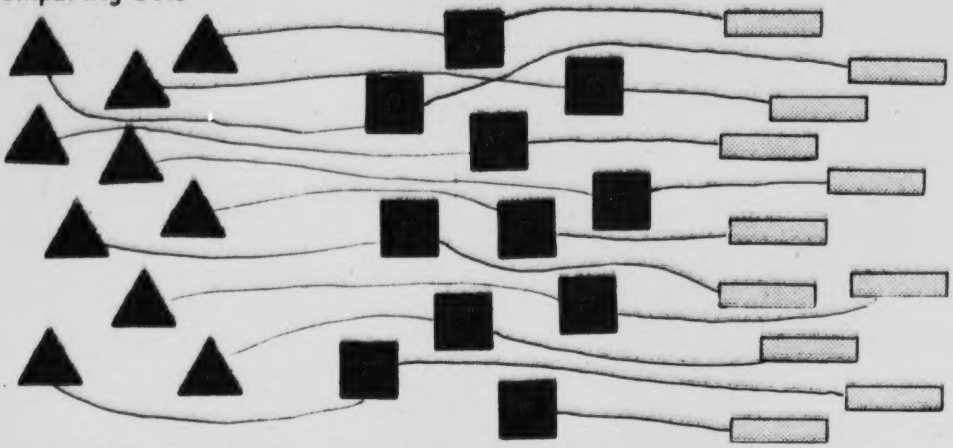
Set B is the set of whole numbers greater than 27 but less than 31.

Set B = {28, 29, 30}

The numbers in the intersection of Sets A and B are: {28, 29}

The symbol ⟷ is used in this teaching exercise to emphasize that 28 and 29 are members common to both A and B. If the symbol (⟷) seems too difficult for children to make, they can encircle the numerals which are the same.

Comparing Sets



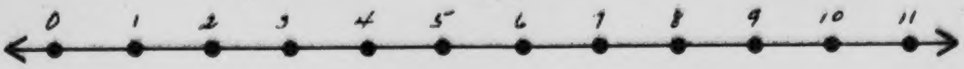
1) There are more squares than △'s .
Show by pairing that your answer is correct. (One □ not paired with a △)

2) There are as many □'s as ▭'s .
Show by pairing that your answer is correct.

3) Is the set of ▲'s equivalent to the set of ■'s? No
How do you know? One less △

4) How many members in the set of ▲'s? 10

5) How many members in the set of ■'s? 11



6) How does the number line help you remember that:
(a) 11 is greater than 10? 11 is to the right of 10
(b) 8 is less than 10? 8 is to the left of 10

Comparing Numbers

Write either < or > between each pair of numerals:

Remember: 7 < 9 is read "7 is less than 9"   9 > 5 is read "9 is greater than 5"

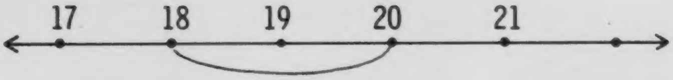
9 < 15
15 < 19
8 > 6
21 > 17
35 > 31
47 < 52
28 > 21
75 < 80
3 < 7
53 < 55

18 > 16
45 < 51
81 > 35
23 > 8
17 < 25
38 < 49
67 > 62
11 < 29
14 < 31
29 < 43

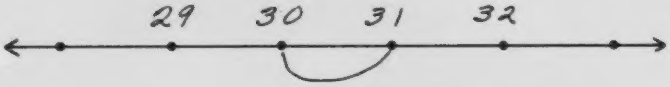
23 < 32
72 > 67
89 > 73
76 < 84
43 > 25
28 < 39
17 < 37
39 > 26
62 > 47
99 < 102

Using the Number Line

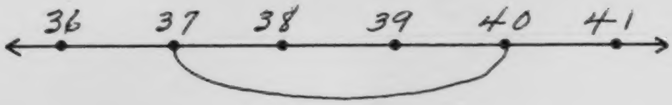
The set of whole numbers greater than 17 but less than 21 is
{18, 19, 20}.


17
18
19
20
21

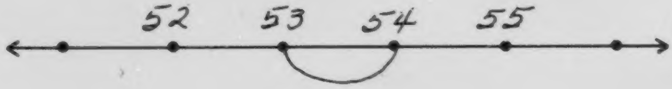
1. The set of whole numbers greater than 29 but less than 32 is
{ 30, 31 }.



2. The set of whole numbers greater than 36 but less than 41 is
{ 37, 38, 39, 40 }.


36
37
38
39
40
41

3. The set of whole numbers greater than 52 but less than 55 is
{ 53, 54 }.



4. The set of whole numbers greater than 92 but less than 88 is
{ }. empty set

Intersection of Sets of Numbers

1. Set A is the set of whole numbers greater than 12 but less than 18.

Set A = {13, 14, 15, 16, 17}

Set B is the set of whole numbers greater than 9 but less than 16.

Set B = {10, 11, 12, 13, 14, 15

The members in the intersection of sets A and B are:

{13, 14, 15}

2. Set R is the set of whole numbers greater than 50 but less than 54.

Set R = {51, 52, 53}

Set T is the set of whole numbers greater than 48 but less than 53.

Set T = {49, 50, 51, 52}

The members in the intersection of sets R and T are:

{51, 52}

3. Set F is the set of whole numbers greater than 47 but less than 53.

Set F = {48, 49, 50, 51, 52}

Set G is the set of whole numbers greater than 50 but less than 57.

Set G = {51, 52, 53, 54, 55 56}

The members in the intersection of sets F and G are:

{51, 52}

4. Set X is the set of whole numbers greater than 79 but less than 85.

Set X = {80, 81, 82, 83, 84}

Set Y is the set of whole numbers greater than 82 but less than 90.

Set Y = {83, 84, 85, 86, 87, 88, 89}

The members in the intersection of sets X and Y are:

{83, 84}

II-3. Addition and subtraction facts.

Objective: To help children achieve mastery of the basic addition and subtraction facts.

Vocabulary: (No new words.)

Materials: (No special materials required.)

Suggested Procedure:

An example of a basic addition fact is

$$\boxed{6 + 3 = 9}$$

Each basic addition fact involves the sum of two whole numbers, each equal to or greater than 0 but less than 10.

To be used quickly and effectively, the basic addition and subtraction facts have to be memorized. However, the memory cannot always be trusted, even if supported by understanding. The process of memorization is enhanced by understanding. If one is unable to recall a basic fact, understanding may provide the means to reconstruct it. Mastery of the basic addition and subtraction facts consists of both understanding and memorization. Satisfactory progress toward such mastery may take significantly more time and effort for some children than for others.

A systematic method of summarizing the basic addition facts is provided by the Addition Fact Chart in the pupil's book, page 59. To facilitate a review, we begin with a limited chart like that below.

Second Addend

First Addend

+	0	1	2	3	4
0					
1					
2					
3			5		
4					

Rows

Columns

Recall with children the way this small chart is used and read. The basic addition fact indicated by the 5 appearing in the chart is

$$3 + 2 = 5$$

because the 5 is in the row labeled 3 (called the 3-row) and in the column labeled 2 (called the 2-column).

If we want to indicate the basic addition fact

$$2 + 3 = 5,$$

we first go to the 2-row and then to the 3-column. In the blank space which is in both of these, we enter 5. Have pupils do the work on their own small charts.

Similar procedures enable us to complete this small table. The following order may be used, although it is not necessary to do so:

4 + 0 = 4
2 + 1 = 3
0 + 2 = 2
3 + 3 = 6
1 + 4 = 5
3 + 1 = 4
1 + 2 = 3
0 + 0 = 0
4 + 1 = 5
2 + 2 = 4
4 + 3 = 7
0 + 3 = 3

0 + 4 = 4
2 + 0 = 2
3 + 4 = 7
0 + 1 = 1
1 + 3 = 4
2 + 4 = 6
3 + 0 = 3
1 + 1 = 2
4 + 4 = 8
1 + 0 = 1
4 + 2 = 6

Have each child complete his chart independently. Note those children who may be encountering difficulty or who may be using immature ways of determining sums--counting on fingers, tapping with a pencil, making dot pictures, etc. Also observe systematic ways that some children may use to complete the chart; e.g., completing the chart by rows, completing the chart by columns, etc. Such observations will enable you to plan appropriately for the needs of individual children as they work toward mastery of the facts.

When the small charts are completed and perhaps at the beginning of a new day's work, attention might be called to "doubles" facts, e.g., $0 + 0 = 0$, $1 + 1 = 2$, $2 + 2 = 4$, $3 + 3 = 6$, $4 + 4 = 8$. When the table is folded along a diagonal running through the plus sign and the 8 in this small chart, all the squares which then fall upon each other contain the same numeral. Thus the 5 indicating $3 + 2 = 5$ falls upon the 5 indicating $2 + 3 = 5$. Similarly, the 5 indicating $1 + 4 = 5$ falls upon the 5 indicating $4 + 1 = 5$. The 7 for $1 + 6 = 7$ falls on the 7 for $6 + 1 = 7$. And so on throughout the chart. This is due to the commutative property of addition.

Children should be encouraged to look for other interesting things about the small table. There are many. For example, the entries just below the heavy line are the same as the labels of the columns in which they appear. This is because $0 + 4 = 4$, and similarly for every other addition fact where 0 is the first number; in such facts, the sum is the same as the second number.

Another interesting thing about this chart is that it can be used to read off the basic subtraction facts. For example, not only is the 5 printed in the chart associated with the addition fact

$$3 + 2 = 5,$$

it also is associated with the subtraction fact

$$5 - 3 = 2.$$

This may be remembered as follows: In a subtraction fact the first number is the sum. Its numeral will be found inside the chart, i.e., to the right of the vertical heavy line and below the horizontal heavy line. Hence to read the fact

$$5 - 3 = 2$$

we go first to the 3-column (3 to be subtracted). We proceed down the 3-column until we find 5 inside the chart. Then we proceed left along the row in which 5 was found until we reach the label of that row. In this case it is 2.

To read the fact

$$6 - 2 = 4,$$

we first go to the 2-column. We proceed down the 2-column until we encounter 6 inside the chart. Then we turn to the left and go until we find the label of the row in which that 6 appears.

Let us agree that the subtraction fact $6 - 2 = 4$ is to be associated with the addition fact $4 + 2 = 6$ while the subtraction fact $6 - 4 = 2$ is to be associated with the addition fact $2 + 4 = 6$.

Then it will be easy to use the completed addition fact chart for reading both addition facts and subtraction facts.

(It is interesting to note that this small chart can be used, without change, as part of a complete addition fact chart for any system of place-value enumeration where the base is nine or more.)

- On another day pupils may be asked to fill in the complete Addition Fact Chart in the pupil's book, page 59. It may be desirable to observe the children in this work to see if they are using any of the knowledge they gained while working with the smaller chart.

When the chart is completed, its uses in addition as well as in subtraction should be demonstrated and practiced in class.

Pupil's book, pages 60-61:

These pages are designed to help children relate subtraction to addition and addition to subtraction.

Pupil's book, pages 62-75: Miscellaneous review.

These pages are illustrations of pupil pages that you may find helpful to use. These illustrations relate to partitions, to using a ten in addition and subtraction, and to miscellaneous addition and subtraction practice. For example, pages devoted to partitions of sets of eight and sets of ten are illustrated. Pages like these can be prepared for pupils who need further experience with partitioning sets with a specified number of members: 12, 15, or whatever. In a similar way other pages may be adapted according to the needs of your particular class.

Addition Chart

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Relating Subtraction to Addition

Fill in the blank. Then write the associated addition fact.

Example: 10 - 4 = 6
6 + 4 = 10

1. 13 - 7 = 6
6 + 7 = 13

2. 11 - 6 = 5
5 + 6 = 11

3. 16 - 8 = 8
8 + 8 = 16

4. 14 - 6 = 8
8 + 6 = 14

5. 17 - 9 = 8
8 + 9 = 17

6. 15 - 8 = 7
7 + 8 = 15

7. 14 - 9 = 5
5 + 9 = 14

8. 12 - 5 = 7
7 + 5 = 12

9. 13 - 8 = 5
8 + 5 = 13

10. 16 - 7 = 9
9 + 7 = 16

Relating Addition to Subtraction

Complete. Then rewrite each addition fact as a subtraction fact.

9 + 8 = 17

17 - 8 = 9

1. 4 + 8 = 12
12 - 8 = 4

2. 7 + 6 = 13
13 - 6 = 7

3. 7 + 4 = 11
11 - 4 = 7

4. 8 + 7 = 15
15 - 7 = 8

5. 5 + 7 = 12
12 - 7 = 5

6. 3 + 9 = 12
12 - 9 = 3

7. 6 + 8 = 14
14 - 8 = 6

8. 9 + 6 = 15
15 - 6 = 9

9. 6 + 5 = 11
11 - 5 = 6

10. 5 + 9 = 14
14 - 9 = 5

11. 4 + 9 = 13
13 - 9 = 4

12. 9 + 9 = 18
18 - 9 = 9

Miscellaneous Exercises
Addition and Related Subtraction

10

4 + 6 = 10
6 + 4 = 10
10 - 6 = 4
10 - 4 = 6

10

2 + 8 = 10
8 + 2 = 10
10 - 8 = 2
10 - 2 = 8

10

3 + 7 = 10
7 + 3 = 10
10 - 3 = 7
10 - 7 = 3

10

9 + 1 = 10
1 + 9 = 10
10 - 1 = 9
10 - 9 = 1

9

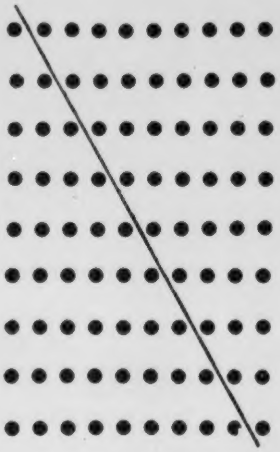
7 + 2 = 9
2 + 7 = 9
9 - 2 = 7
9 - 2 = 7

9

5 + 4 = 9
4 + 5 = 9
9 - 4 = 5
9 - 5 = 4

Partitions of a Set of Ten Things

Write an equation for each row.



10 = 1 + _____

10 = 2 + _____

10 = 3 + _____

10 = 4 + _____

10 = 5 + _____

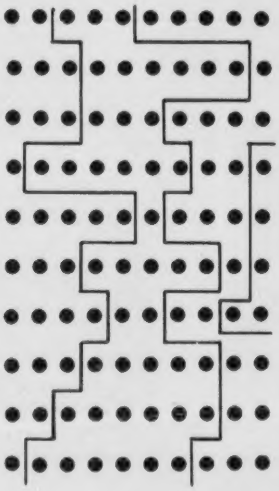
10 = _____ + _____

10 = _____ + 3

10 = _____ + _____

10 = _____ + _____

Write an equation for each row.



2 + 3 + 5 = 10

3 + 5 + 1 = 10

3 + 3 + 4 = 10

1 + 6 + 2 + 1 = 10

5 + 1 + 3 + 1 = 10

3 + 5 + 1 + 1 = 10

4 + 2 + 2 + 2 = 10

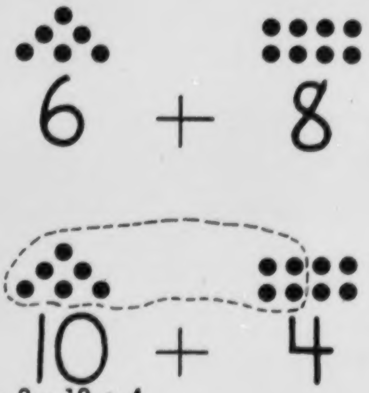
3 + 5 + 2 = 10

2 + 6 + 2 = 10

1 + 6 + 3 = 10

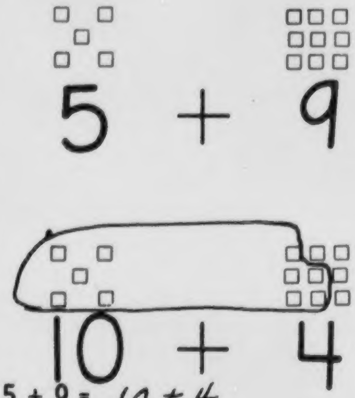
Using a Ten in Addition and Subtraction

Join some of the members of the second set to the first set to make a group of ten.



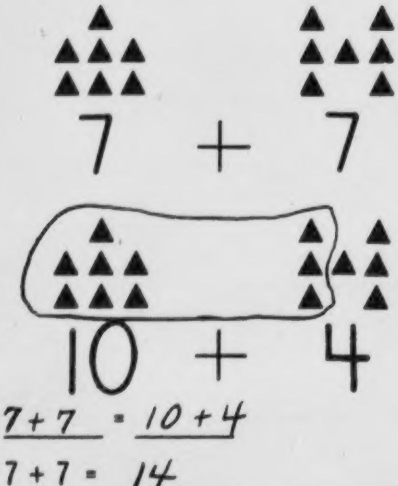
6 + 8 = 10 + 4

6 + 8 = 14



5 + 9 = 10 + 4

5 + 9 = 14



7+7 = 10+4

7 + 7 = 14

FINISH:

9 + 1 + 4 = 14

8 + 2 + 4 = 14

7 + 3 + 4 = 14

6 + 6 + 2 = 14

5 + 5 + 4 = 14

4 + 6 + 4 = 14

3 + 7 + 4 = 14

2 + 8 + 4 = 14

1 + 9 + 4 = 14

Using a Ten in Addition and Subtraction

Think of the sum of the two numbers as 10 and some ones.

6 + 7 = 10 + 3

9 + 5 = 10 + 4

7 + 4 = 10 + 1

8 + 6 = 10 + 4

9 + 6 = 10 + 5

5 + 6 = 10 + 1

8 + 5 = 10 + 3

4 + 8 = 10 + 2

9 + 2 = 10 + 1

8 + 8 = 10 + 6

8 + 4 = 10 + 2

7 + 5 = 10 + 2

5 + 9 = 10 + 4

8 + 3 = 10 + 1

6 + 8 = 10 + 4

7 + 6 = 10 + 3

9 + 4 = 10 + 3

9 + 3 = 10 + 2

5 + 8 = 10 + 3

4 + 7 = 10 + 1

6 + 5 = 10 + 1

9 + 9 = 10 + 8

8 + 9 = 10 + 7

7 + 7 = 10 + 4

Pairs of Numbers

Complete this chart.

Number Pair First number	 Second number	Operation	Whole number sum or difference
7	4	+	11
12	3	-	9
6	10	-	Not any
5	8	+	13
15	9	-	6
8	7	+	15
7	14	-	Not any
16	8	-	8
5	9	+	14
9	5	—	4
5	9	—	Not any
6	0	+	6
12	12	—	0

Miscellaneous Exercises

Write two addends for each sum. Then change the order of the addends. Use numbers less than 10.

11	12	13	14
9, 2	9, 3	9, 4	9, 5
2, 9	3, 9	4, 9	5, 9
8, 3	8, 4	8, 5	8, 6
3, 8	4, 8	5, 8	6, 8
7, 4	7, 5	7, 6	7, 7
4, 7	5, 7	6, 7	
6, 5	6, 6		
5, 6			

15	16	17	18
9, 6	8, 8	9, 8	9, 9
6, 9	9, 7	8, 9	
7, 8	7, 9		
8, 7			

Miscellaneous Exercises

The sum of two numbers is named in each larger box. Below each sum is one of the addends. Name the other addend. The first one is done for you.

12	
5	7
8	4
4	8

11	
3	8
9	2
7	4

13	
6	7
9	4
7	6

14	
9	5
8	6
6	8

13	
5	8
7	6
4	9

16	
7	9
8	8
9	7

12	
10	2
9	3
7	5

15	
9	6
8	7
4	11

Finish each equation.

8 + 7 = 15 15 + 7 = 22	8 + 8 = 16 16 - 8 = 8
6 + 9 = 15 15 + 9 = 24	9 + 5 = 14 14 - 5 = 9
9 + 8 = 17 17 + 8 = 25	6 + 8 = 14 14 - 8 = 6
7 + 9 = 16 16 - 9 = 7	8 + 6 = 14 14 - 6 = 8
5 + 8 = 13 8 + 5 = 13	5 + 9 = 14 9 + 5 = 14
5 + 7 = 12 7 + 5 = 12	8 + 9 = 17 9 + 8 = 17

Miscellaneous Exercises

Fill in the blanks so that in each row the sum of the first two numbers is the third number

and

in each column the sum of the first two numbers is the third number.

(4)	2	6
0	3	3
4	5	9

1	3	4
6	0	6
7	3	10

5	3	8
2	2	4
7	5	12

4	0	4
4	5	9
8	5	13

3	5	8
2	1	3
5	6	11

3	4	7
5	2	7
8	6	14

Miscellaneous Exercises

Fill in the charts by finding the sum of pairs of numbers.

	2	7	6	8
5	7	12	11	13
9	11	16	15	17
3	5	10	9	11
4	6	11	10	12

	7	5	9	8
6	13	11	15	14
0	7	5	9	8
8	15	13	17	16
5	12	10	14	13

	4	6	7	9
5	9	11	12	14
1	5	7	8	10
6	10	12	13	15
7	11	13	14	16

	8	6	9	5
8	16	14	17	13
5	13	11	14	10
7	15	13	16	12
6	14	12	15	11

Miscellaneous Exercises

Make these sentences true by using = , <, or >.

1. 7 + 6 --=-- 6 + 7
2. 5 + 8 --<-- 5 + 9
3. 9 + 3 -->-- 2 + 9
4. 6 + 3 --<-- 9 + 1
5. 6 + 5 --=-- 5 + 6
6. 2 + 9 --=-- 9 + 2
7. 7 + 3 --<-- 4 + 7
8. 6 + 6 -->-- 6 + 5
9. 2 + 7 --<-- 3 + 7
10. 4 + 8 --=-- 8 + 4
11. 6 + 2 --=-- 2 + 6
12. 2 + 9 --=-- 8 + 3
13. 5 + 8 --=-- 8 + 5
14. 4 + 8 -->-- 6 + 5
15. 3 + 9 --=-- 9 + 3
16. 6 + 4 --=-- 7 + 3

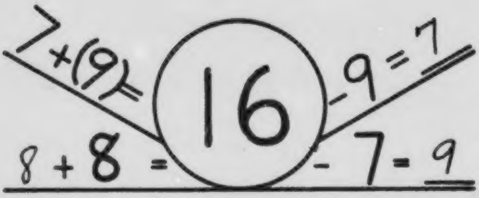
Make these true by using + and - .

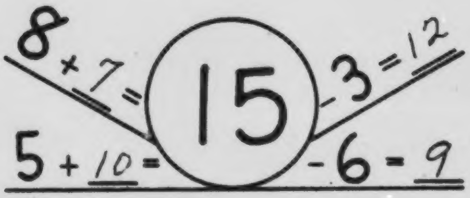
1. 4 -+- 2 > 7 ---- 5
2. 9 ---- 7 < 8 ---- 4
3. 8 ---- 2 < 7 -+- 2
4. 7 -+- 7 > 8 -+- 2
5. 8 -+- 9 < 9 -+- 9
6. 9 ---- 4 > 7 ---- 6
7. 14 ---- 3 > 8 ---- 4
8. 9 ---- 8 < 6 ---- 4
9. 17 ---- 2 > 8 -+- 4
10. 21 ---- 6 < 9 -+- 8
11. 28 ---- 4 > 15 -+- 8
12. 34 ---- 7 < 25 -+- 15
13. 79 -+- 24 > 149 ---- 57
14. 56 ---- 29 < 12 -+- 14
15. 89 -+- 45 > 134 ---- 51
16. 201 -+- 98 > 300 ---- 56

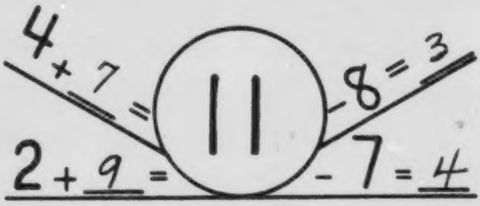
Miscellaneous Exercises

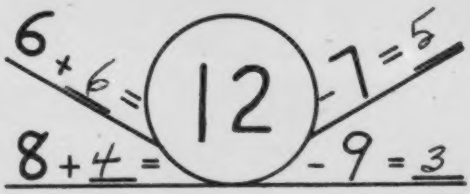
Fill in the blanks with the correct numerals.

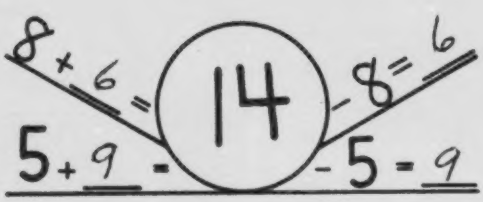
Begin at the left and go clockwise.

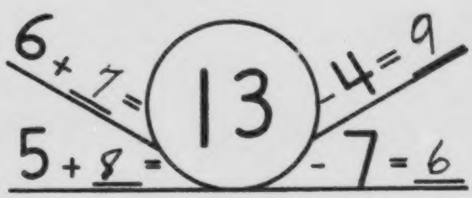

7+(9)=
16
-9=7
8 + 8 =
- 7 = 9


8 + 7 =
15
-3 = 12
5 + 10 =
-6 = 9


4 + 7 =
11
-8 = 3
2 + 9 =
- 7 = 4


6 + 6 =
12
-7 = 5
8 + 4 =
- 9 = 3


8 + 6 =
14
-8 = 6
5 + 9 =
-5 = 9

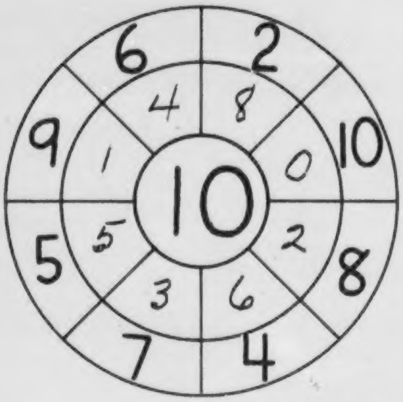

6 + 7 =
13
-4 = 9
5 + 8 =
- 7 = 6

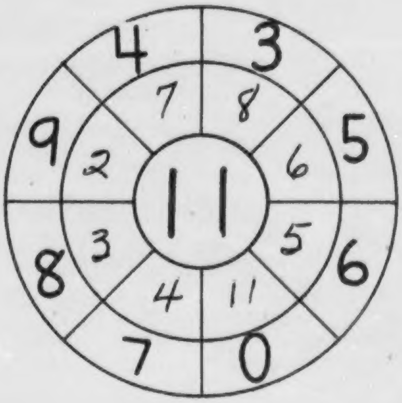
Miscellaneous Exercises

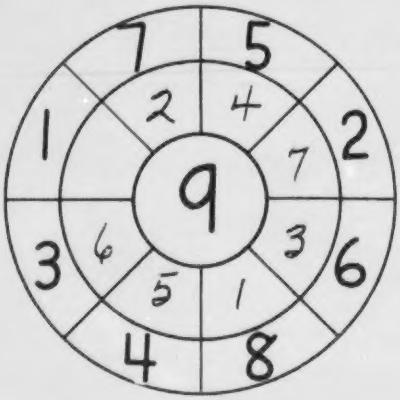
Fill in the second ring.

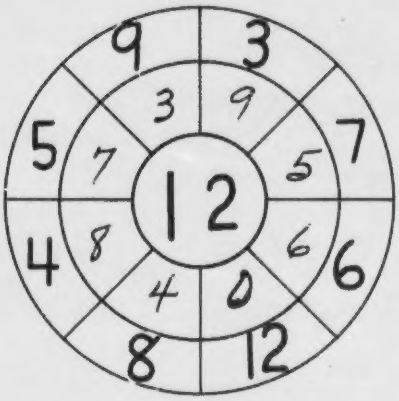
Given addend plus other addend equals the sum named in the third ring. Example: $2 + n = 13$

$n = ?$


6
2
10
8
4
7
5
9
4
8
0
2
6
3
5
1
10

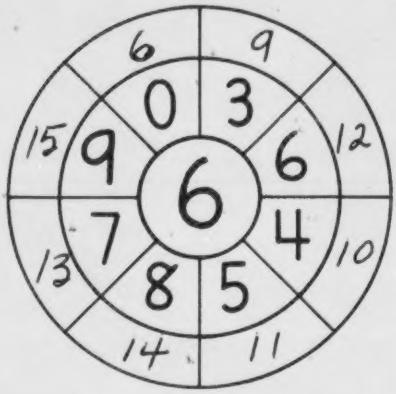

4
3
5
6
0
7
8
9
7
8
6
5
11
4
3
2
11

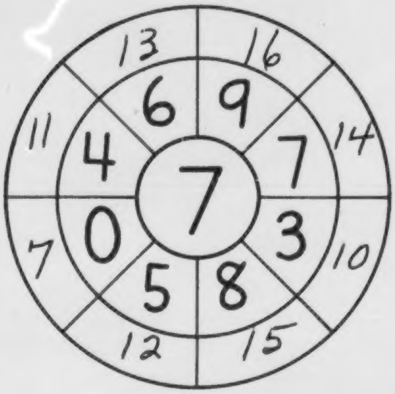

7
5
2
6
8
4
3
1
2
4
7
3
1
5
6
9


9
3
7
6
12
8
4
5
3
9
5
6
0
4
8
7
12

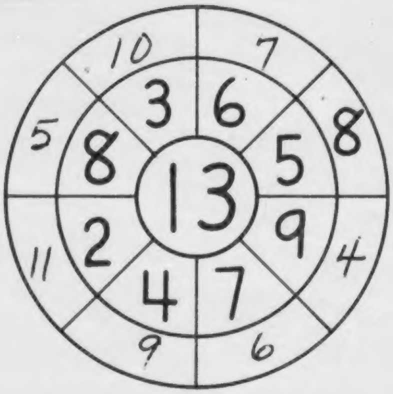
Miscellaneous Exercises

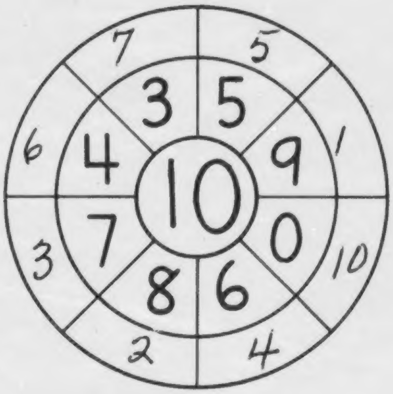
1. Find the sum by adding the number named in the center ring to a number named in the second ring. Write the sum in the outer ring.


6
0
3
6
4
5
8
7
9
6
9
12
10
11
14
13
15


7
6
9
7
3
8
5
0
4
13
16
14
10
15
12
7
11

2. Find the difference by subtracting a number named in the second ring from the number named in the center ring. For example: 13 - 5 = 8.


13
3
6
5
9
7
4
2
8
10
7
8
4
6
9
11
5


10
3
5
9
0
6
8
7
4
7
5
1
10
4
2
3
6

II-4. Place value

Objective: To extend the idea of place value and the decimal base of our numeration system.

Vocabulary: (No new words.)

Materials: Dowel rods (or similar materials) for forming sets of hundreds, tens and ones. (These may or may not be needed with the children in your class.)

Abacuses for teacher and pupil use.

Suggested Procedure:

This section of work might be introduced with the following "make-believe" newspaper clipping. If possible, duplicate copies and distribute one to each child.

> Last Saturday 836 people visited the Central City zoo. There were 683 children among the visitors. At noon, lunches were served in the lunch room to 386 of the visitors.

Ask the following questions. Record on the chalkboard the numeral for each answer:

How many people visited the zoo? (836.)
How many of them were children? (683.)
How many noon lunches were served? (386.)

Lead to the fact that the same digits--3, 6, and 8--appear in each numeral. Then have children react to this assertion:

> Since the same digits appear in each numeral, the three numerals are just different names for the same number! So, all the people who visited the zoo were children, and all the people who visited the zoo ate lunch in the lunch room at noon.

Children's reactions naturally should lead to the fact that the position or place of a digit in a numeral is very important--that 836 and 683 and 368 are names for different numbers; that the 8 in 836 refers to 8 hundreds or 800, that the 8 in 683 refers to 8 tens or 80, and that the 8 in 368 refers to 8 ones or simply 8. Similar things may be done for the digits 3 and 6.

Write the following on the chalkboard and see if children are able to complete each correctly and quickly:

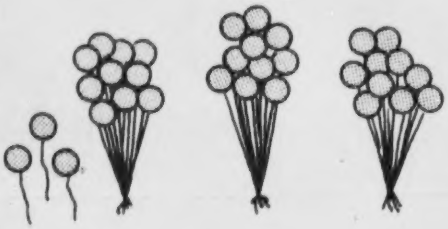
6 hundreds, 8 tens, 3 ones or _____
8 hundreds, 3 tens, 6 ones or _____
3 hundreds, 6 tens, 8 ones or _____

800 + 30 + 6 = _____
300 + 60 + 8 = _____
600 + 80 + 3 = _____

The preceding explorations will give you an indication of the way in which you profitably may use pages 76 - 79 in the pupil's book to extend the children's understanding of place value. For some children the first two or three pages may be merely sampled or even eliminated. For other children you will wish to have them work carefully through each of these four pages.

Tens and Ones

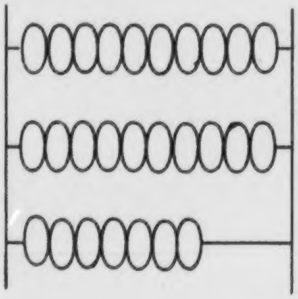
Fill in the blanks.



3 tens, 3 ones

or

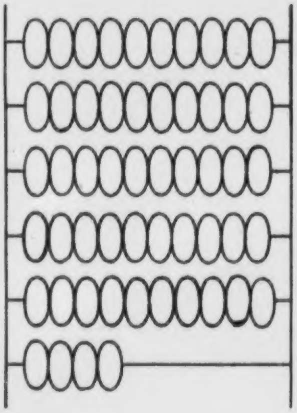
33



2 tens, 7 ones

or

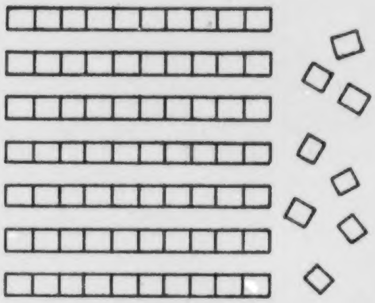
27



5 tens, 4 ones

or

54



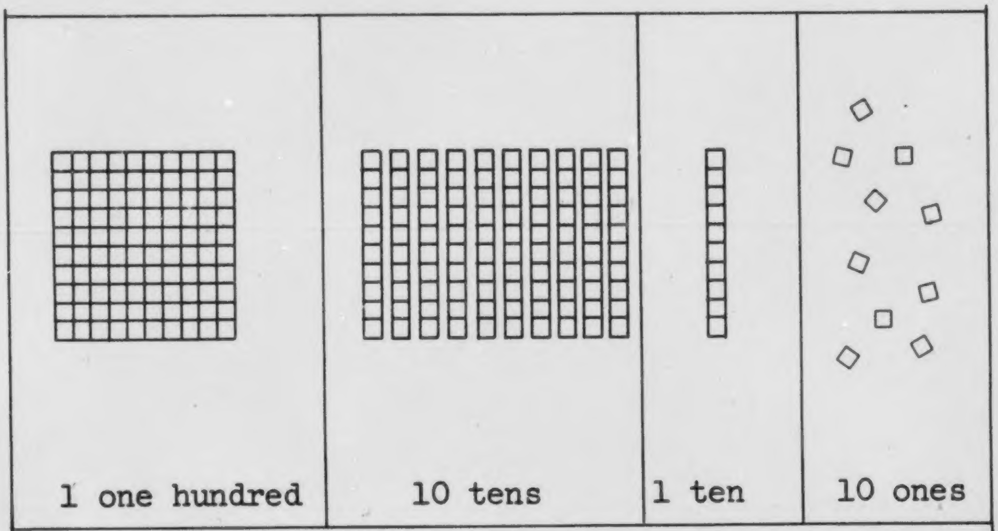
7 tens, 8 ones

or

78

Hundreds, Tens, and Ones

Complete each of these.



186: 1 hundred, 8 tens, and 6 ones

342: 3 hundreds, 4 tens, and 2 ones

203: 2 hundreds, 0 tens, and 3 ones

230: 2 hundreds, 3 tens, and 0 ones

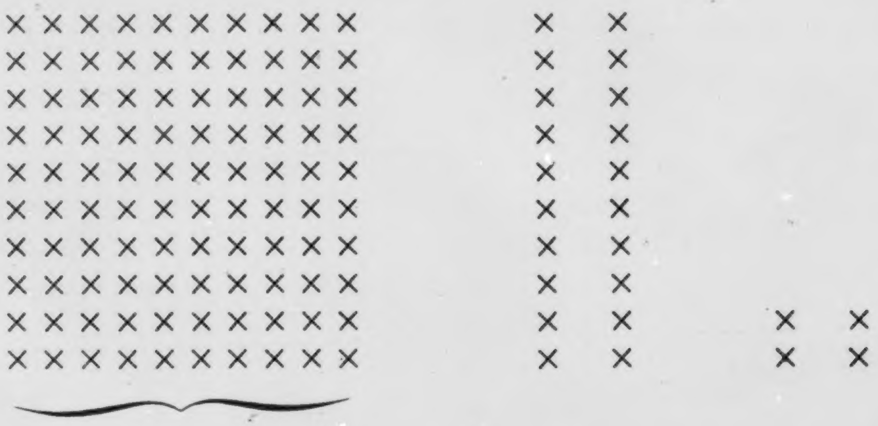
625: 6 hundreds, 2 tens, and 5 ones

496: 4 hundreds, 9 tens, and 6 ones

704: 7 hundreds, 0 tens, and 4 ones

541: 5 hundreds, 4 tens, and 1 one

Hundreds, Tens, and Ones



124 = 100 + 20 + 4

Fill in the blanks:

563 = 500 + 60 + 3

247 = 200 + 40 + 7

486 = 400 + 80 + 6

625 = 600 + 20 + 5

741 = 700 + 40 + 1

598 = 500 + 90 + 8

460 = 400 + 60

802 = 800 + 2

Hundreds, Tens, and Ones

Fill the blanks.

847 = 800 + 40 + 7

235 = 200 + 30 + 5

670 = 600 + 70 + 0

509 = 500 + 0 + 9

999 = 900 + 90 + 9

419 = 400 + 10 + 9

704 = 700 + 4

128 = 20 + 100 + 8

836 = 6 + 30 + 800

280 = 200 + 80

605 = 5 + 600

603 = 600 + 0 + 3

476 = 400 + 70 + 6

875 = 800 + 70 + 5

570 = 500 + 70 + 0

409 = 400 + 0 + 9

888 = 800 + 8 + 8

810 = 800 + 10

235 = 200 + 30 + 5

604 = 4 + 600

980 = 80 + 900

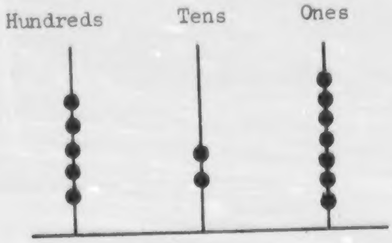
205 = 5 + 200

- The Abacus

It is appropriate at this time to introduce children to the abacus as a device for representing numbers.

Using a numeral such as 527, have it represented as 5 sets of 100, 2 sets of 10, and 7 ones. Then develop the notion that if we use the idea of place value, we merely need to show the number of hundreds (in the appropriate place), the number of tens (in the appropriate place), and the number of ones (in the appropriate place).

Introduce an abacus with three rods. Make clear which rod is to be associated with the hundreds' place, with the tens' place, and with the ones' place. Lead children to see that the following representation is a valid one for 527.



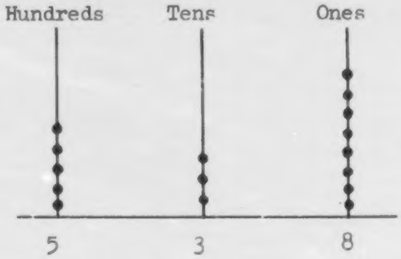
Make clear that we can replace 10 beads on the ones rod (10 ones) with one bead on the tens rod (1 ten), and that we can replace 10 beads on the tens rod (10 tens) with one bead on the hundreds rod (1 hundred).

Provide experiences in which children are asked to: (a) represent a given 3-place numeral on the abacus, and (b) tell what numeral has been represented on an abacus. Use page 80 in the pupil's book in developing these understandings.

Place Value

An abacus can help us represent a number.

1.

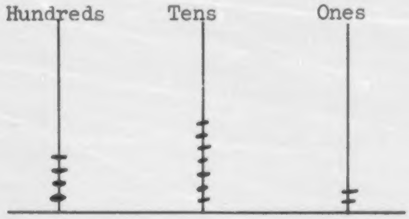

Hundreds
Tens
Ones
5
3
8

The number 538 is represented on this abacus.

538 = 5 hundreds + 3 tens + 8 ones, or

538 = 500 + 30 + 8.

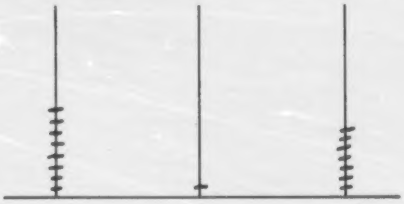
2.


Hundreds
Tens
Ones

Show 472 on this abacus.

472 = 400 + 70 + 2

3.



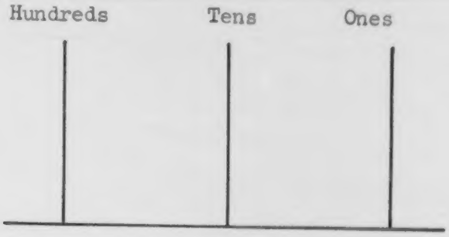
Show 817 on this abacus.

817 = 800 + 10 + 7

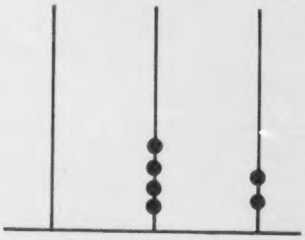
Regrouping sets and renaming numbers

This work related to regrouping sets and renaming numbers is important in preparation for future work in addition and subtraction. Children need to have experiences which help them realize that a number may be represented in different ways. This can be shown on an abacus.

Draw an abacus on the chalkboard:

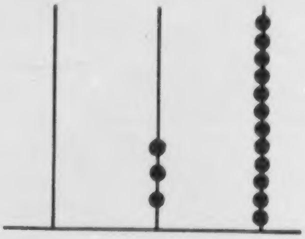


Ask children how 42 would be shown on the abacus; then, how 42 can be shown with only 3 tens.



4 tens and 2 ones

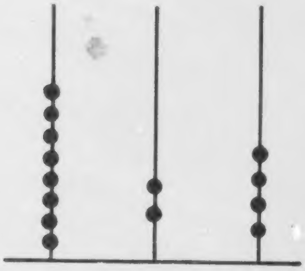
40 + 2 = 42



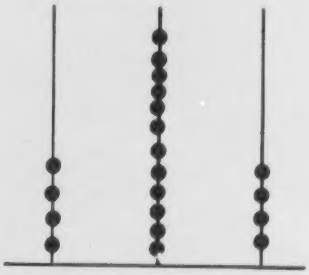
3 tens and 12 ones

30 + 12 = 42

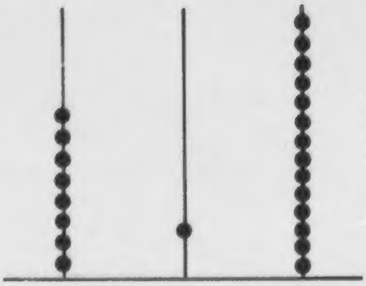
Ask children how 824 can be shown on the abacus; then, how 824 can be shown with only 7 hundreds; then, how it can be shown with only one ten.



800 + 20 + 4 = 824



700 + 120 + 4 = 824

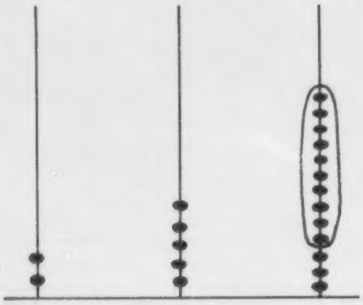


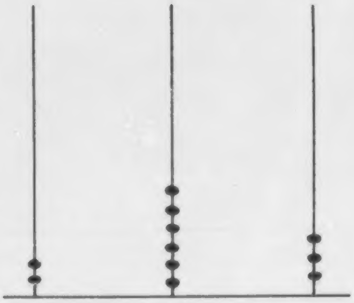
800 + 10 + 14 = 824

Pupil's book, pages 81 - 91 Give additional practice in renaming numbers.

Renaming Numbers

1.





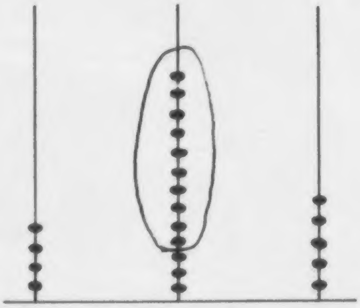
200 + 50 + 12 = 263

A set of ten ones can be shown as

one ten.

200 + 60 + 3 = 263

2.



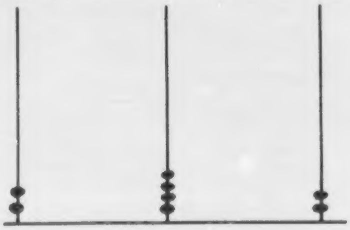
400 + 130 + 5 = 535

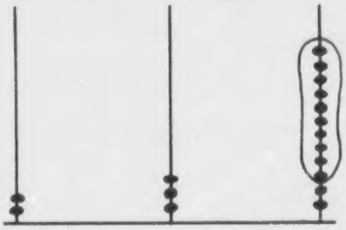
A set of ten tens can be shown as

one hundred.

Renaming Numbers

3.



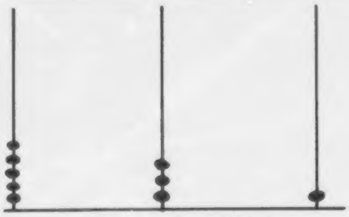


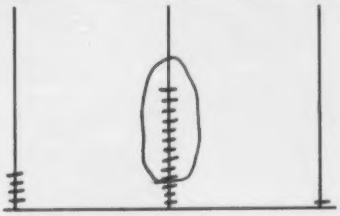
200 + 40 + 2 = 242

Show one set of ten as a set of ten ones.

Write the new name. 200 + 30 + 12

4.



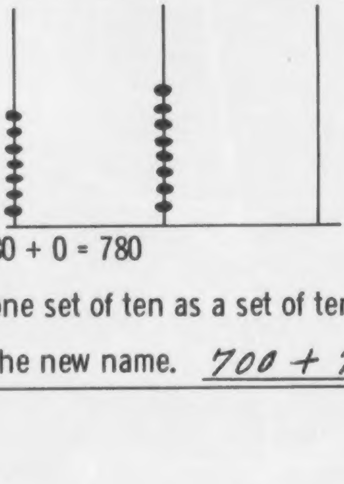


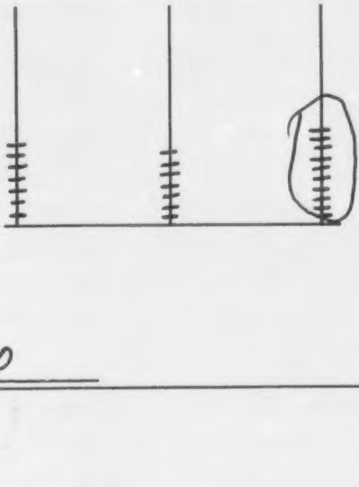
500 + 30 + 1 = 531

Show one hundred as a set of ten tens.

Write the new name. 400 + 130 + 1

5.



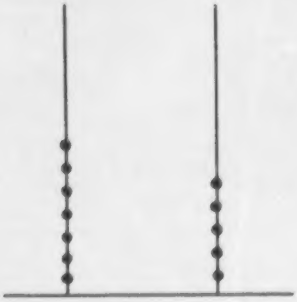


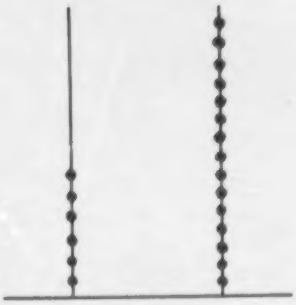
700 + 80 + 0 = 780

Show one set of ten as a set of ten ones.

Write the new name. 700 + 70 + 10

Different Ways of Thinking About a Number





75 = 7 tens + 5 ones = 6 tens + 15 ones

68 = 6 tens + 8 ones
or 5 tens + 18 ones

94 = 9 tens + 4 ones
or 8 tens + 14 ones

39 = 3 tens + 9 ones
or 2 tens + 19 ones

62 = 6 tens + 2 ones
or 5 tens + 12 ones

49 = 4 tens + 9 ones
or 3 tens + 19 ones

57 = 5 tens + 7 ones
or 4 tens + 17 ones

84 = 8 tens + 4 ones
or 7 tens + 14 ones

71 = 7 tens + 1 one
or 6 tens + 11 ones

96 = 9 tens + 6 ones
or 8 tens + 16 ones

74 = 7 tens + 4 ones
or 6 tens + 14 ones

Naming a Number in Different Ways

Complete the following sentences.

357 = 3 hundreds + 5 tens + 7 ones,
or 3 hundreds + 4 tens + 17 ones,
or 2 hundreds + 14 tens + 17 ones.

268 = 2 hundreds + 6 tens + 8 ones,
or 2 hundreds + 5 tens + 18 ones,
or 1 hundred + 15 tens + 18 ones.

569 = 56 tens + 9 ones,
or 4 hundreds + 16 tens + 9 ones,
or 4 hundreds + 15 tens + 19 ones.

Write 426 in three other ways.

4 hundreds + 2 tens + 6 ones
4 hundreds + 1 ten + 16 ones
3 hundreds + 11 tens + 16 ones

Write 752 in three other ways.

7 hundreds + 5 tens + 2 ones
7 hundreds + 4 tens + 12 ones
6 hundreds + 14 tens + 12 ones

Renaming a Number

Match the expanded form with the standard form. For example, (A) 100 + 40 + 3 = 143, so A is placed in the blank beside 143.

A	100 + 40 + 3	I	400 + 90 + 1
B	500 + 70 + 12	J	600 + 10 + 15
C	600 + 160 + 4	K	500 + 80 + 2
D	900 + 20 + 2	L	700 + 00 + 16
E	300 + 00 + 7	M	700 + 60 + 4
F	600 + 110 + 6	N	200 + 100 + 7
G	100 + 30 + 13	O	500 + 120 + 5
H	200 + 10 + 17	P	800 + 120 + 2

764 M

491 I

716 L

227 H

143 A

922 D

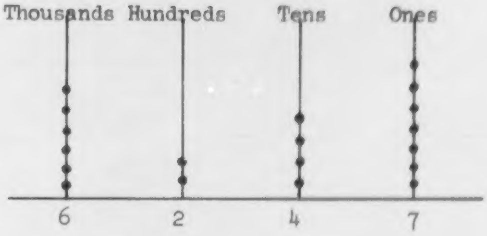
582 B

625 J

307 E

Thousands

1.

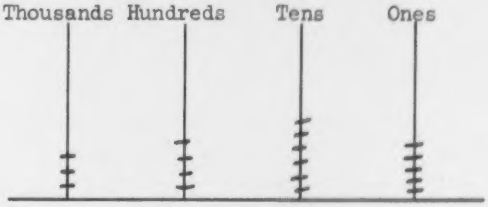

Thousands
Hundreds
Tens
Ones
6
2
4
7

The number 6247 is represented on this abacus.

6247 = 6 thousands + 2 hundreds + 4 tens + 7 ones

6247 = 6000 + 200 + 40 + 7

2. Show 3465 on this abacus.

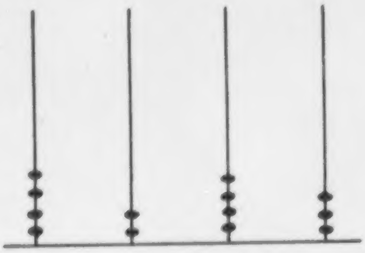

Thousands
Hundreds
Tens
Ones

3465 = 3 thousands + 4 hundreds + 6 tens + 5 ones

3465 = 3000 + 400 + 60 + 5

Renaming Numbers

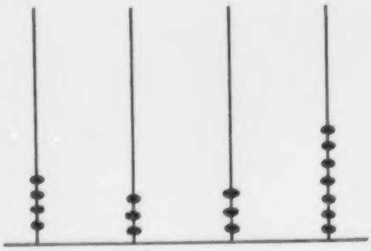
1.



4000 + 200 + 40 + 3 =

4243

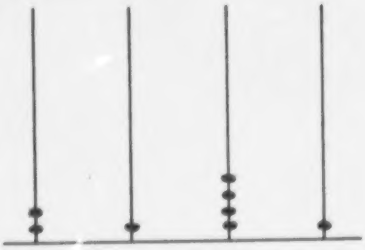
2.



6000 + 300 + 30 + 7 =

6337

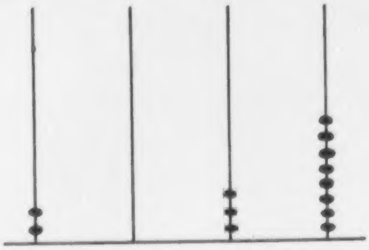
3.



2000 + 100 + 40 + 1 =

2141

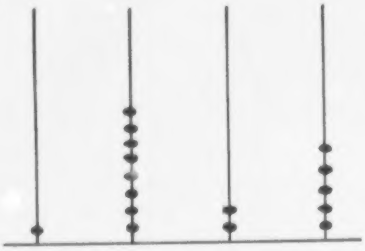
4.



2000 + 000 + 30 + 8 =

2038

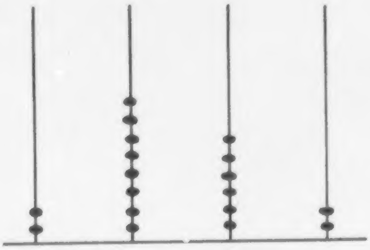
5.



1000 + 800 + 20 + 5 =

1825

6.



2000 + 800 + 60 + 2 =

2862

Thousands

Complete each of these:

10 ones = 1 ten

10 tens = 1 hundred

10 hundreds = 1 thousand

2748 = 2 thousands + 7 hundreds + 4 tens + 8 ones

5619 = 5 thousands + 6 hundreds + 1 ten + 9 ones

7546 = 7 thousands + 5 hundreds + 4 tens + 6 ones

5380 = 5 thousands + 3 hundreds + 8 tens + 0 ones

3074 = 3 thousands + 0 hundreds + 7 tens + 4 ones

9206 = 9 thousands + 2 hundreds + 0 tens + 6 ones

6324 = 6000 + 300 + 20 + 4

5289 = 5000 + 200 + 80 + 9

9165 = 9000 + 100 + 60 + 5

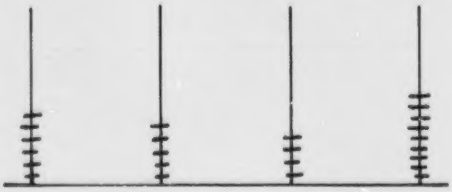
2912 = 2000 + 900 + 10 + 2

7503 = 7000 + 500 + 3

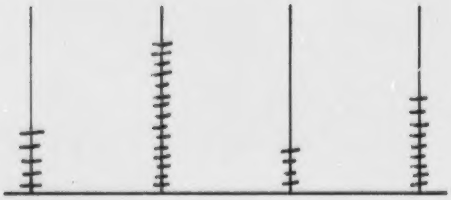
4087 = 4000 + 80 + 7

Naming a Number in Different Ways

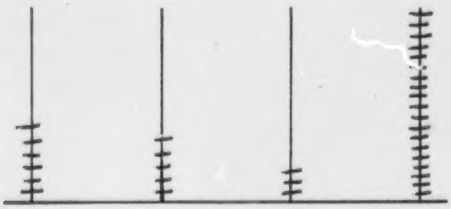
1. Show 6549 on the abacus with 6 thousands, 5 hundreds, 4 tens and 9 ones.



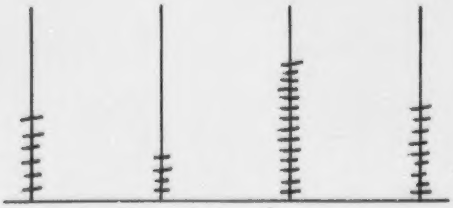
2. Show 6549 with only 5 thousands.



3. Show 6549 with only 3 tens.



4. Show 6549 with only 4 hundreds.



Naming a Number in Different Ways

1. Here are some ways to name 3547.

3547 ones

35 hundreds + 4 tens + 7 ones

3 thousands + 5 hundreds + 4 tens + 7 ones

354 tens + 7 ones

3000 + 500 + 40 + 7

3500 + 40 + 7

2. Show some ways to name 2356.

2356 ones

23 hundreds + 5 tens + 6 ones

2 thousands + 3 hundreds + 5 tens + 6 ones

235 tens and 6 ones

2000 + 300 + 50 + 6

2300 + 50 + 6

3. Show some ways to name 4253.

4253 ones

42 hundreds + 5 tens + 3 ones

4 thousands + 2 hundreds + 5 tens + 3 ones

425 tens + 3 ones

4000 + 200 + 50 + 3

4200 + 50 + 3

Names for Numbers

1. From the list below check (✓) all the ways of naming 6529.

✓ a) 6,529 ones
✓ b) 652 tens + nine ones
c) 6000 + 500 + 10 + 9
d) 6000 + 1500 + 20 + 9
✓ e) 5000 + 1500 + 20 + 9
✓ f) 65 hundreds + 20 + 9
g) 6000 + 400 + 20 + 9
h) 6000 + 500 + 20 + 19

2. Answer Yes or No.

a) 5,324 is 53 tens and 24 ones. No
b) 7381 = 600 + 120 + 8. No
c) 32 hundreds + 2 tens + 16 ones = 3236. Yes
d) 537 = 400 + 13 + 7. No

3. The number 2, 538 can be named in many ways. Write some of them.

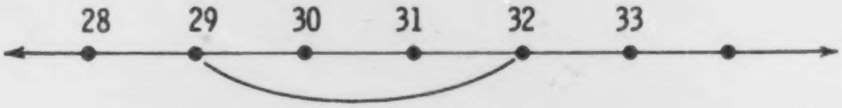
2,538: 2538 ones
2000 + 500 + 30 + 8
2500 + 30 + 8
253 tens + 8 ones

Pupil's book, pages 92-93:

Are concerned with the inequality of numbers. Review with the children the meaning of the symbols > and <. Show them how they may use the number lines to decide which numbers are members of the set greater than one number but less than another. You may wish to use the first example in this discussion observing that the starting point is 28 and the last point named is 33.

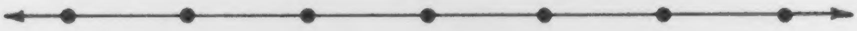
Using the Number Line

The set of whole numbers greater than 28 but less than 33 is {29, 30, 31, 32}.



1. The set of whole numbers greater than 67 but less than 73 is
{68, 69, 70, 71, 72 }.

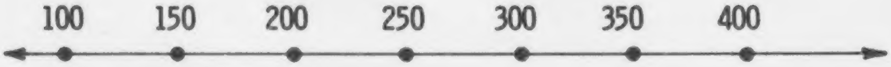
2. The set of whole numbers greater than 198 but less than 204 is
{ }.



3. The set of whole numbers greater than 789 but less than 800 is
{ }.

4. The set of whole numbers greater than 993 but less than 1002 is
{ }.

Comparing Numbers


100
150
200
250
300
350
400

Write < or > between each pair of numerals.

129 < 156

175 < 200

235 > 167

253 < 350

391 < 450

402 > 343

156 < 380

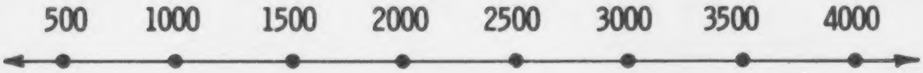
287 < 459

376 > 285

491 > 176

207 < 377

176 < 253


500
1000
1500
2000
2500
3000
3500
4000

500 < 1500

2000 > 1000

3500 > 2500

4000 > 500

3520 > 2001

756 < 1156

2356 < 2556

3702 > 3046

3427 < 3548

2763 > 3276

4051 > 4027

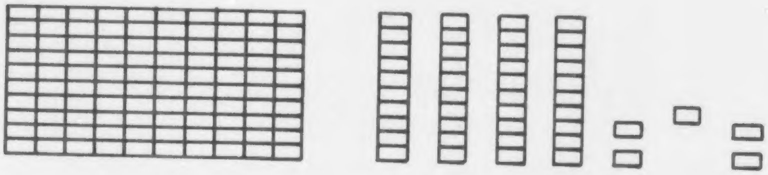
1776 > 1492

II-5. Techniques for finding sums

Objective: To review techniques for adding numbers between 10 and 100 using expanded forms, and to extend those techniques to adding numbers between 100 and 1000.

Vocabulary: (No new words.)

Materials: Sticks or other materials that can be grouped easily into sets of hundreds, tens, and single units such as the hundred card, ten cards, and single unit cards below:



Suggested Procedure:

The material in the first part of this section is intended as a quick review of addition techniques. 35 + 24 could be computed in the following ways:

$$\begin{array}{r} 30 + 5 \\ \underline{+ (20 + 4)} \\ 50 + 9 = 59 \end{array} \qquad \begin{array}{r} 35 \\ \underline{+ 24} \\ 9 \\ \underline{+ 50} \\ 59 \end{array}$$

If children have not used these forms previously, then use developmental materials in Book 2 and develop these techniques as other ways to compute the sum. This review might be motivated by using a problem situation.

Display 3 packages of tickets (10 tickets in each package) and 5 single tickets without telling how many are in the set.

Here are some tickets.

Who thinks he can tell us how many we have? (See if children observe sets of tens and ones.)

Now suppose we put another set of tickets with this first set. Here is our second set: 2 tens and 4 ones.

Who can tell us how many tickets are in it? (24)

How many tickets do we have in the union of the two sets? (35 + 24.)

Now who can show us how to find another name for this number?

Children may suggest several ways. If no child suggests the expanded form, then you suggest it by starting to solve the problem and asking someone to try to complete the work. For example,

I am thinking of another way we might use. Suppose we think of 35 as 30 + 5. Then how would we think of 24 in the same way? (20 + 4)

Now let us write

$$\begin{array}{r} 30 + 5 \\ \underline{20 + 4} \end{array}$$

Who can finish the problem? (Hopefully, some child will do so. Be willing to let some try, even unsuccessfully.)

I am thinking of still another way. I will start it.

$$\begin{array}{r} 35 \\ \underline{24} \\ 50 \end{array}$$

Now, who will finish the work this time?

Have children perform other such computations. Select from the following:

23 + 35 = ____   22 + 37 = ____   73 + 22 = ____

46 + 31 = ____   15 + 13 = ____   56 + 43 = ____

73 + 15 = ____   42 + 17 = ____   64 + 24 = ____

Ask if these sums were more difficult to compute than sums which involve just basic combination (i.e., 8 + 7 = __, 9 + 6 = __). After some discussion the children should realize that these numbers greater than 10 are not more difficult, since our numeration system makes it as easy to add tens as to add ones.

Further Activities

Suggest that children think of these computations as describing problems like the ticket problem. For instance, some children may wish to suggest problem situations that could be described by 46 + 31 = 77.

> There are 46 children playing.
> Then 31 children came to play.
> How many children are playing?

Pupil's book, pages 94 - 95: Use these pages for independent work. Since both expanded forms have been reviewed, allow the children to select the form best suited to their needs. Children who have the understanding and no longer need the expanded form should be permitted to use the regular algorithm.

The Sum of Two Numbers

Compute:

52 + 37 Ans. 89	83 + 16 Ans. 99
26 + 42 Ans. 68	43 + 55 Ans. 98
72 + 13 Ans. 85	14 + 44 Ans. 58

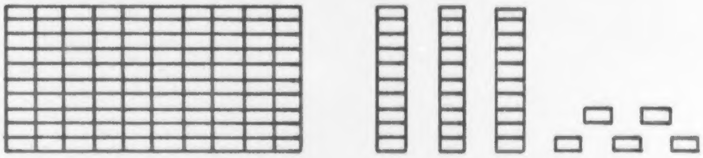
The Sum of Two Numbers

Compute:

67 + 32 Ans. 99	45 + 56 Ans. 101
74 + 15 Ans. 89	58 + 31 Ans. 89
46 + 53 Ans. 99	36 + 32 Ans. 68

- Extending to finding sums of numbers greater than 100

Now let us think about many more tickets. Display a set of 135 objects as 1 set of one hundred, 3 sets of ten, and 5 ones. (You may find materials as illustrated below easier to use.)



Then present another set of your own choice, e.g., a set of 163 objects, so that the sum can be found without renaming ones, tens, or hundreds. Ask children to tell how many are in each set (135 and 163). Then ask how they can find the number of members in the union of the two sets (add 135 and 163). Write:

$$135 + 163 = \underline{\qquad}.$$

Recall from previous work on place value how these numbers can be expressed. Questions asked should elicit answers that will result in the following records:

$$\begin{array}{rlr} 135 & 100 + 30 + 5 & 135 \\ \underline{+\ 163} & \underline{(100 + 60 + 3)} & \underline{+\ 163} \\ & 200 + 90 + 8 \ = 298 & 8 \\ & & 90 \\ & & \underline{200} \\ & & 298 \end{array}$$

$$135 + 163 = 298.$$

Find sums of other pairs of numbers where each number is greater than 100. Use both techniques for several examples unless, for individual children, this does not seem advisable. Then let children use the form which has most meaning according to their ability and skill. You may wish to extend to beyond 1000 for some children. Talk about the similarity of adding these numbers and adding numbers less than 100.

173 + 225 = ___   375 + 622 = ___   1654 + 5235 = ___
542 + 354 = ___   670 + 308 = ___   3075 + 3903 = ___
304 + 495 = ___   831 + 168 = ___   3407 + 3582 = ___
254 + 441 = ___   532 + 463 = ___   5176 + 2723 = ___

Use Pupil's book, pages 96 - 97: For independent practice.

The Sum of Two Numbers

Compute:

362 + 507 Ans. 869	450 + 249 Ans. 699
743 + 253 Ans. 996	804 + 194 Ans. 998
512 + 466 Ans. 978	277 + 702 Ans. 979

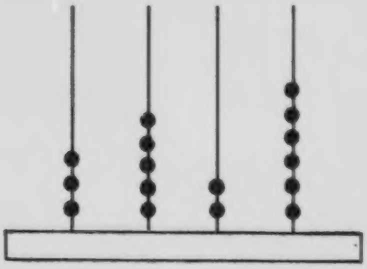
The Sum of Two Numbers

Compute:

127 + 651 Ans. 778	504 + 265 Ans. 769
1645 + 8253 Ans. 9898	7064 + 1825 Ans. 8889
8403 + 1596 Ans. 9999	3754 + 5005 Ans. 8759

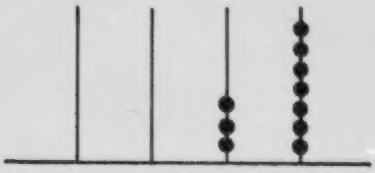
- Using the abacus

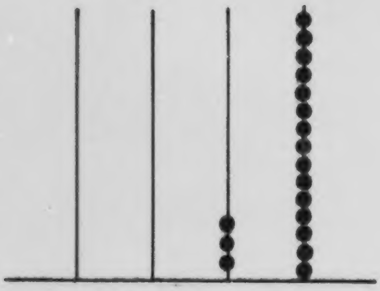
Children are familiar with the abacus from their work on place value. Review by asking what number is represented, such as,



representing the number 3,526.

Select problems according to the particular abilities of your group. You may wish to start with the second illustration rather than the first one that is suggested.

> We can use the abacus to help us represent numbers when we have problems to solve. We can use it instead of sticks, tickets (or whatever material was being used).
>
> Suppose 37 doughnuts were served at a party, but 8 more doughnuts were needed if everyone was to be served.
>
> How many doughnuts were needed for the party? Who can write this sum?
>
> 37 + 8
>
> Who thinks he can represent the number of doughnuts served first?
>
> 
>
> You thought of it as 3 tens and 7 ones.
>
> How many more were needed? (8)



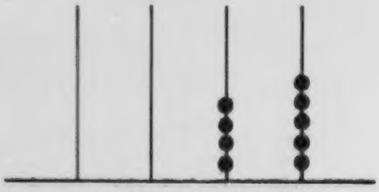
Let us make a record of what we have done.

$$\begin{array}{r} 3 \text{ tens} + 7 \text{ ones} \\ \underline{+ 8 \text{ ones}} \\ 3 \text{ tens} + 15 \text{ ones} \end{array}$$

Can we think of 15 ones in another way? (Yes.)

15 ones = 1 ten + 5 ones

We can move 10 counters from the ones rod and place one more counter on the tens rod.



Now we must change the written record.

$$\begin{array}{l} 3 \text{ tens} + 7 \text{ ones} \\ \underline{\qquad\quad + 8 \text{ ones}} \\ 3 \text{ tens} + 15 \text{ ones} = 4 \text{ tens} + 5 \text{ ones} \end{array}$$

Let's see if we can find a shorter way to compute sums.

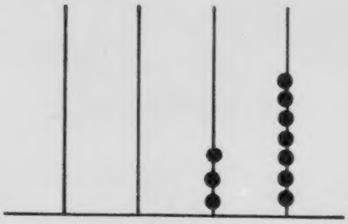
You may select one of the following (or use both).

$$\begin{array}{r} 37 \\ \underline{8} \end{array} \qquad \begin{array}{l} 30 + 7 \\ \underline{(\quad + 8)} \\ 30 + 15 = 40 + 5 = 45 \end{array} \qquad \begin{array}{r} 37 \\ \underline{+\ 8} \\ 15 \\ \underline{30} \\ 45 \end{array}$$

If both are used, encourage children to describe the differences and likenesses.

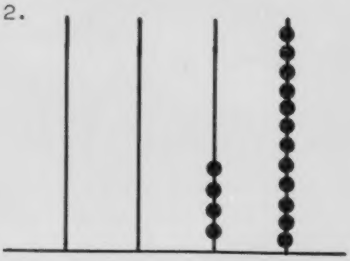
- Continue with another problem. You may wish to keep it in the context of party, doughnuts, etc. Instead of needing 8 more doughnuts, 15 more are needed. After stating the problem, ask a child to give the example which describes the problem (37 + 15 = ___). Call on a child to represent the numbers on the abacus and do the regrouping. Ask two other children to come to the board and keep a record of each step as it is taken.

1.



37 = 30 + 7

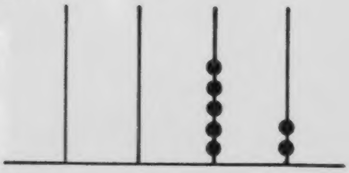
2.



37 = 30 + 7

15 = 10 + 5

3.



$$\begin{array}{rl} 37 = 30 + 7 & \\ \underline{15 = 10 + 5} & \\ 40 + 12 = & \\ 50 + 2 = 52 & \end{array} \qquad \begin{array}{r} 37 \\ \underline{15} \\ 12 \\ \underline{40} \\ 52 \end{array}$$

Write sums such as these on the board.

36 + 27 43 + 55

See if the children can tell just by looking if the ones and tens must be renamed. Thinking carefully before putting pencil to paper will pay big dividends when working with expanded notation in the solution of subtraction examples requiring renaming.

Pupil's book, pages 98 - 103: Discuss the first two pages with the children. Use the others for independent work.

Pupil's book. pages 104 - 110: The next four pages deal with the addition of numbers greater than 99 but less than 1000. The only renaming necessary has already been taught. You may use the first page as your introduction or you may prefer to begin with a problem situation.

> Last year our school had 549 books in the library.
> This year we received 324 more.
> How many books do we have now?

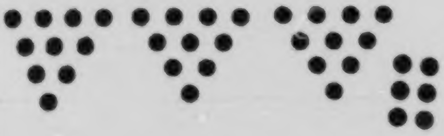
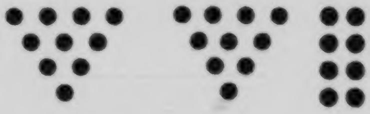
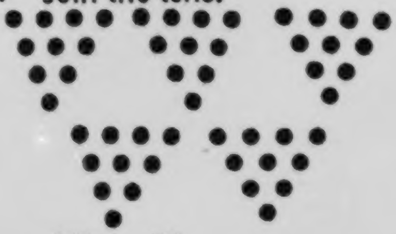
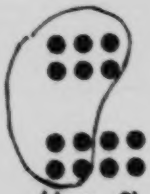
The procedures suggested for the use of the abacus on page 159 of the Teachers' Commentary can be used in this situation. Then the first pupil page can be discussed with the children.

Use the remaining pages for independent work.

The Sum of Two Numbers

Mary has a bouquet with 36 flowers. If Jill gives her a bouquet having 28 flowers, how many flowers will Mary have? We may write:

36 + 28 = n ______

A. Think of 36 as:  (30 + 6)	B. Think of 28 as:  (20 + 8)
C. Join the tens:  (30 + 20)	D. Join the ones:  (6 + 8)

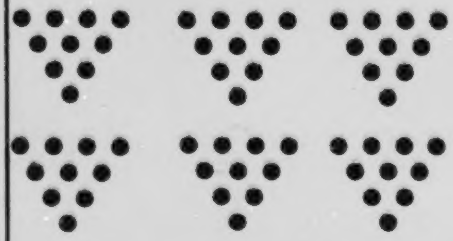
Do you see that we have another set of ten when we join the ones?

Make a ring around a set of ten.

6 + 8 = 10 + 4 ______.

(30 + 20) + (10 + 4) = n ______.

E. Join the new set of ten to the other sets of ten.



(30 + 20 + 10)

These are the ones.

(4)

$$(30 + 20 + 10) + 4 = 60 + 4 = 64$$

F. You can write:

$$\begin{array}{l} 36 + 28 \\ 36 = 30 + 6 \\ 28 = \underline{20 + 8} \\ \quad\; 50 + 14 = 50 + 10 + 4 = 64 \end{array}$$

OR

$$\begin{array}{r} 36 \\ + \underline{28} \\ 14 \\ \underline{50} \\ 64 \end{array}$$

Renaming Ones

Mark those for which you would rename 10 ones as 1 ten.

1) 27 + 35 ✓

2) 57 + 26 ✓

3) 54 + 25

4) 73 + 27 ✓

5) 41 + 14

6) 43 + 26

7) 35 + 40

8) 26 + 38 ✓

9) 37 + 48 ✓

10) 74 + 13

11) 29 + 8 ✓

12) 25 + 18 ✓

13) 45 + 9 ✓

14) 42 + 56

15) 67 + 23 ✓

16) 57 + 16 ✓

17) 34 + 57 ✓

18) 23 + 64

19) 89 + 7 ✓

20) 66 + 27 ✓

21) 47 + 29 ✓

22) 28 + 39 ✓

23) 33 + 52

24) 17 + 64 ✓

The Sum of Two Numbers

Compute:

63 + 29 Ans. 92	58 + 25 Ans. 83
54 + 27 Ans. 81	49 + 28 Ans. 77
65 + 29 Ans. 94	23 + 47 Ans. 70

The Sum of Two Numbers

Compute:

76 + 18 Ans. 94	67 + 19 Ans. 86
58 + 7 Ans. 65	59 + 38 Ans. 97
35 + 46 Ans. 81	47 + 9 Ans. 56

The Sum of Two Numbers

Compute:

1) 59 + 37 = 96

2) 46 + 28 = 74

3) 37 + 55 = 92

4) 14 + 78 = 92

5) 25 + 69 = 94

6) 38 + 47 = 85

7) 65 + 26 = 91

8) 47 + 37 = 84

9) 63 + 19 = 82

10) 54 + 37 = 91

11) 63 + 28 = 91

12) 15 + 75 = 90

13) 39 + 59 = 98

14) 28 + 69 = 97

15) 47 + 39 = 86

16) 29 + 28 = 57

The Sum of Two Numbers

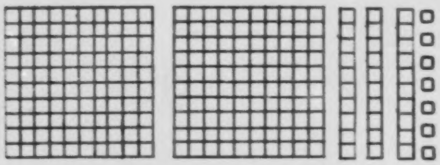
Ann has 237 stamps in her stamp collection.

Her grandmother gave her 126 more stamps.

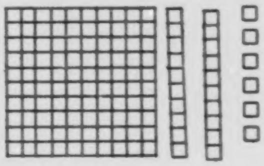
How many stamps does Ann have now?

We write: 237 + 126

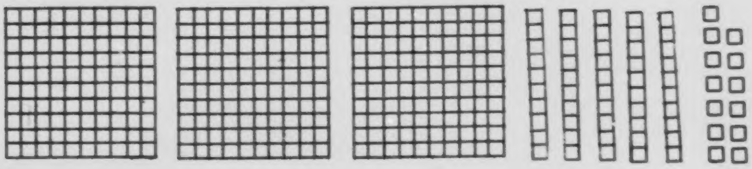
Think of 237 as:



Think of 126 as:



Join the hundreds, then the tens, and then the ones.



Think of 13 as 10 + 3.

So, 237 + 126 = 300 + 50 + 13

= 300 + 60 + 3

= 363

We can write:

$$\begin{array}{rcl} 237 & = & 200 + 30 + 7 \\ 126 & = & \underline{100 + 20 + 6} \\ & & 300 + 50 + 13 = 300 + 60 + 3 = 363 \end{array}$$

OR

$$\begin{array}{r} 237 \\ +\ \underline{126} \\ 13 \\ 50 \\ \underline{300} \\ 363 \end{array}$$

Ann has 363 stamps.

The Sum of Two Numbers

Compute:

345 +249 Ans. 594	538 +237 Ans. 775
816 +185 Ans. 1001	248 + 125 Ans. 373
347 +226 Ans. 573	723 +158 Ans. 881
707 +105 Ans. 812	349 +233 Ans. 582

The Sum of Two Numbers

Compute:

248 +129 Ans. 377	394 +283 Ans. 677
369 +128 Ans. 497	348 +161 Ans. 509
764 + 29 Ans. 793	586 +123 Ans. 709
459 + 26 Ans. 485	340 +360 Ans. 700

The Sum of Two Numbers

Compute:

204 + 567 Ans. 771

348 + 236 Ans. 574

753 + 239 Ans. 992

546 + 329 Ans. 875

728 + 267 Ans. 995

806 + 187 Ans. 993

The Sum of Two Numbers

Compute:

437 + 243 Ans. 680	461 + 279 Ans. 740
537 + 256 Ans. 793	825 + 137 Ans. 962
347 + 268 Ans. 615	158 + 629 Ans. 787

The Sum of Two Numbers

Compute:

1) 532 + 149 = 681

2) 304 + 177 = 481

3) 348 + 29 = 377

4) 502 + 378 = 880

5) 37 + 156 = 193

6) 848 + 129 = 977

7) 325 + 39 = 364

8) 207 + 308 = 515

9) 206 + 385 = 591

10) 81 + 19 = 100

11) 469 + 317 = 786

12) 36 + 407 = 443

13) 409 + 217 = 626

14) 268 + 206 = 474

15) 74 + 16 = 90

16) 67 + 208 = 275

17) 146 + 726 = 872

18) 848 + 108 = 956

19) 37 + 207 = 244

20) 475 + 206 = 681

21) 671 + 329 = 1000

22) 106 + 87 = 193

23) 164 + 206 = 370

24) 129 + 69 = 198

A Vacation Trip

1. Ed's parents took him to visit a park.
They drove 269 miles the first day.
The second day they went 317 miles.
How far did they travel in 2 days? (586 miles)

2. Ed saw 14 different car license plates.
The next day he saw 9.
He claims he saw 24 in 2 days.
Did he? How do you know? (No. 23 < 24)

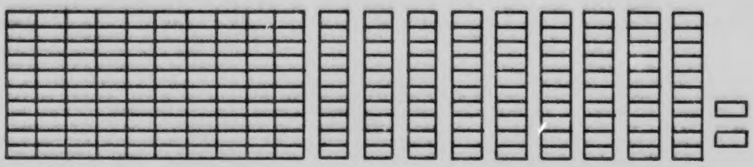
3. On Monday 406 cars went into the park.
On Tuesday 375 more came in.
How many visited the park on Monday and Tuesday? (781)

4. There were 14 bears and 8 deer along the road.
Ed saw them.
How many animals did he see? (22)

5. Ed ate $6.38 worth of food.
His share of the motel bill was $3.38.
What did his trip cost his father? ($9.76)

- **Renaming 10 tens as one hundred**

Display a set of 192 objects as one set of one hundred, 9 sets of ten and 2 sets of one. (You may find material as illustrated below easier to use.)



Then present another set of your own choice, e.g., 186. Ask children to tell how many are in each set (192 and 186). Also ask how they can find the number of members in the union of the two sets. Write:

192 + 186 = ____.

Recall how these numbers can be expressed by referring to previous work on place value. Questions asked should result in the following records.

$$\begin{array}{ll} 192 & 100 + 90 + 2 \\ \underline{186} & \underline{100 + 80 + 6} \\ & 200 + 170 + 8 = 300 + 70 + 8 \\ & \qquad\qquad\quad\;\; = 378 \end{array}$$

$$\begin{array}{r} 192 \\ \underline{+\ 186} \\ 8 \\ 170 \\ \underline{200} \\ 378 \end{array}$$

and in answering the question of how many objects are in the union of the two sets.

Find sums of other pairs of numbers where each number is greater than 100 and renaming ten tens as one hundred is required.

Use both techniques for several examples, unless for individual children this does not seem advisable. Then let children use the form of their choice in keeping with their level of understanding and skill.

The use of the abacus as illustrated in the Teachers' Commentary page 159 can be adapted for this type of regrouping example.

Pupil's book, pages 111 - 116 Discuss the first page with the children. The following pages are for independent work.

Finding the Sum of Two Numbers

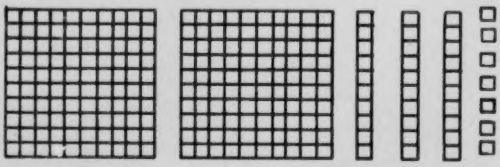
Ann had 237 stamps in her collection.

Her grandmother gave her 191 more stamps.

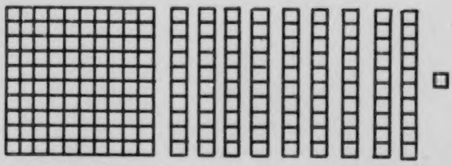
How many stamps does Ann have now?

We write: 237 + 191

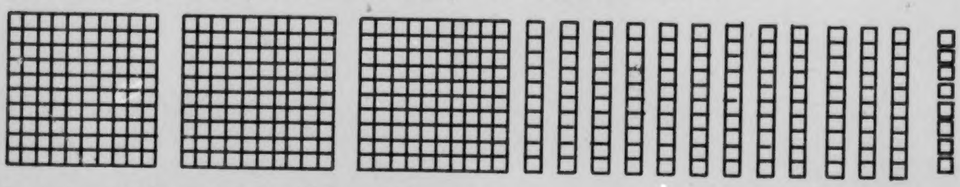
Think of 237 as:



Think of 191 as:



Join the hundreds, then the tens, and then the ones.



Think of 12 tens as 100 + 20.

$$\begin{aligned} \text{So,}\quad 237 + 191 &= 300 + 120 + 8 \\ &= 300 + 100 + 20 + 8 \\ &= 400 + 20 + 8 \\ &= 428 \end{aligned}$$

We can write:

$$\begin{array}{rcl} 237 &=& 200 + 30 + 7 \\ 191 &=& \underline{100 + 90 + 1} \\ && 300 + 120 + 8 \\ && 400 + 20 + 8 = 428 \end{array}$$

OR

$$\begin{array}{r} 237 \\ +\underline{191} \\ 8 \\ 120 \\ \underline{300} \\ 428 \end{array}$$

Ann has 428 stamps.

Renaming Ten Tens

1) 300 + 170 + 8
400 + 70 + 8 = 478

2) 500 + 120 + 7
600 + 20 + 7 = 627

3) 100 + 140 + 6
200 + 40 + 6 = 246

4) 700 + 150 + 0
800 + 50 + 0 = 850

5) 200 + 100 + 8
300 + 0 + 8 = 308

6) 400 + 190 + 2
500 + 90 + 2 = 592

7) 800 + 130 + 3
900 + 30 + 3 = 933

8) 600 + 160 + 6
700 + 60 + 6 = 766

9) 100 + 100 + 5
200 + 0 + 5 = 205

10) 800 + 190 + 9
900 + 90 + 9 = 999

Renaming Ten Tens

396 +283 679	765 +173 938
493 +215 708	398 +261 659
613 + 196 809	384 +263 647
794 +173 967	342 +166 508

Renaming Ten Tens

783 + 643 = 1426

495 + 192 = 687

496 + 213 = 709

384 + 571 = 955

764 + 142 = 906

135 + 284 = 419

431 + 176 = 607

327 + 292 = 619

Renaming Ten Tens

Compute the sum.

395 + 282 = 677

784 + 192 = 976

651 + 263 = 914

493 + 276 = 769

364 + 273 = 637

487 + 161 = 648

276 + 550 = 826

386 + 253 = 639

Uncle Jim's Farm

1. Uncle Jim lives 170 miles from Boys' Town.
   Boys' Town is 268 miles from White City.
   Uncle Jim drove to White City by way of Boys' Town.
   How many miles did he travel? (438 miles)

2. Jane visited the farm.
   She saw 76 cows along the highway.
   Uncle Jim has many horses.
   She counted 52.
   Did she see more than 100 animals?
   (Yes. 76+52 = 128)

3. On the farm are 784 hens.
   There are 20 roosters.
   How many chickens does Uncle Jim have? (804)

4. Last year Uncle Jim made $475 in wheat.
   The corn crop was worth $450.
   How much money did he make on grain? ($ 925.00)

5. The hired man put 170 bales of hay in the barn.
   He did the same thing the next week.
   How many bales of hay did he store? (340)

II-6. Techniques for finding differences

Objective: To review techniques for subtracting numbers between 10 and 100, using the expanded form, and to extend those techniques to subtracting numbers between 100 and 1000.

Vocabulary: (No new words.)

Materials: (Same as used in preceding section.)

Suggested Procedure:

As for addition, the material in this section is intended as a quick review of subtraction techniques. $63 - 28 = \_\_$ could be computed in the following ways:

1)

$$\begin{aligned} 63 - 28 &= (60 + 3) - (20 + 8) \\ 63 - 28 &= (50 + 13) - (20 + 8) \\ 63 - 28 &= (50 - 20) + (13 - 8) \\ 63 - 28 &= 30 + 5 = 35 \end{aligned}$$

2)

$$\begin{array}{cccc} 63 & 60 + 3 & 50 + 13 & \\ \underline{-\ 28} & \underline{-\ (20 + 8)} & \underline{-\ (20 + \ 8)} & \\ & & 30 + 5 & = 35 \end{array}$$

The first form is appropriate when working with concrete materials. The second form is more convenient when a written record of procedure is being made. If children do not use the second form because of lack of experience, then use developmental materials in Book 2 as was done for addition.

Motivate the review by presenting a problem situation.

Tom has 48 stamps.
There were 4 blocks of 10 and 8 single stamps.
Who can represent Tom's stamps using these cards? (Or whatever materials selected.)
He used 17 stamps. How many stamps were left?

What should be written on the chalkboard?
Who thinks he knows how to perform the computation?

If no child suggests the expanded form, then begin by starting to solve the problem as indicated below.

$$\begin{array}{rl} 48 & 40 + 8 \\ \underline{-\ 17} & \underline{-(10 + 7)} \\ & 30 + 1 \quad = \quad 31 \end{array}$$

Continue with another problem. You may wish to keep in the context of stamps.

Mary had 63 stamps.
How many blocks of ten stamps each? (6)
How many single stamps? (3)
She used 28 stamps. How many stamps were left?

Ask a child what should be written on the chalkboard. Then ask another child or several children to tell how they would do the computation.

For example:

$$\begin{array}{rcc} 63 & 60 + 3 & 50 + 13 \\ \underline{-\ 28} & \underline{-(20 + 8)} & \underline{-(20 + \ 8)} \\ & & 30 + \ 5 = 35 \end{array}$$

If children have difficulty in thinking of regrouping 6 tens and 3 ones as 5 tens and 13 ones, use concrete materials to represent the stamps. This will make it clear that it is necessary to think of 63 as 50 + 13. Ask how this result differs from the first problem.

- Solve other problems. In every case ask the children if it is necessary to rename a ten as ten ones to perform the computation. They should recognize that this is necessary if the ones digit in the sum is less than the ones digit in the known addend.

Select from the examples given below. Children should look at the example and, if possible, rename the sum in the most efficient form before beginning to work. Given: 41 - 27 = ___, an inspection of the ones digit makes it apparent that the renaming of 41 should be 30 + 11 not 40 + 1. Of course, it is necessary to allow for individual differences.

64 - 27 = ___   67 - 38 = ___   53 - 38 = ___
98 - 24 = ___   57 - 25 = ___   37 - 15 = ___
56 - 29 = ___   92 - 48 = ___   74 - 59 = ___

Pupil's book, pages 117 - 122: The first page should be discussed with the class. Use the others for independent study.

Renaming the Sum

1) 93 - 48
93 = 80 + 13

2) 47 - 19
47 = 30 + 17

3) 54 - 28
54 = 40 + 14

4) 63 - 27
63 = 50 + 13

5) 97 - 19
97 = 80 + 17

6) 55 - 26
55 = 40 + 15

7) 74 - 56
74 = 60 + 14

8) 21 - 17
21 = 11 + 10

9) 36 - 18
36 = 20 + 16

10) 95 - 27
95 = 80 + 15

11) 71 - 38
71 = 60 + 11

12) 65 - 48
65 = 50 + 15

13) 44 - 19
44 = 30 + 14

14) 52 - 39
52 = 40 + 12

Computing the Difference Between Two Numbers

75 - 28 = 47 75 = 60 + 15 - 28 = (20 + 8) 40 + 7 = 47	68 - 29 = 37 68 - 29
84 - 16 = 68 84 - 16	46 - 27 = 19 46 - 27
53 - 24 = 29 53 - 24	35 - 17 = 18 35 - 17
92 - 65 = 27 92 - 65	62 - 48 = 14 62 - 48

Computing the Difference

92 - 85 = 7

94 - 76 = 18

56 - 39 = 17

75 - 58 = 17

25 - 17 = 8

86 - 29 = 57

Computing the Difference

75 - 39 = 36

53 - 34 = 19

64 - 18 = 46

63 - 17 = 46

82 - 24 = 58

81 - 27 = 54

Finding the Difference Between Two Numbers

46 and 19 Ans. 27	43 and 25 Ans. 18
92 and 47 Ans. 45	62 and 44 Ans. 18
53 and 26 Ans. 27	51 and 26 Ans. 25
84 and 35 Ans. 49	67 and 39 Ans. 28
74 and 39 Ans. 35	45 and 16 Ans. 29
82 and 25 Ans. 57	52 and 19 Ans. 33

The Birthday Party

1) Bill invited 35 children to his party.
Yesterday his mother bought a package of balloons.
There were 18 balloons in the package.
Bill wants to give each child a balloon.
How many more ballons does he need ? (17)

2) There are 50 candles in a box.
Bill is 8 years old.
How many candles will not be used? (42)

3) Bill received 29 gifts.
How many children did not bring a gift? (6)

4) John brought Bill a box of marbles.
Bill had 56 marbles.
Now he has 94.
How many marbles were in the box? (38)

5) There were 19 boys at the party.
How many girls were there? (16)

• Extending to numbers greater than 100

We now think about more stamps. Display a set of 185 stamps as 1 set of 100, 8 sets of 10, and 5 sets of one. Ask children to tell the number of stamps in the set. Then suggest:

> Suppose we use 45 of these stamps.
> How many stamps will be left?

Write on the chalkboard, 185 - 45. Ask a child to show how we can work with the set of stamps to solve the problem. Then ask if someone can show how we can solve the problem in another way, using what is written on the board. If they do not suggest the expanded form, you suggest it by starting the problem and letting someone complete it. For example,

$$\begin{array}{r} 185 \\ \underline{-\ 45} \end{array} \qquad \begin{array}{r} 100 + 80 + 5 \\ -\ \underline{\qquad (40 + 5)} \\ 100 + 40 + 0 = 140 \end{array} .$$

Change the situation to a set of 394 stamps from which 173 are removed.

$$\begin{array}{r} 394 \\ \underline{-\ 173} \end{array} \qquad \begin{array}{r} 300 + 90 + 4 \\ \underline{-\ (100 + 70 + 3)} \\ 200 + 20 + 1 \end{array}$$

Find differences for other pairs of numbers greater than 100. You may wish to extend to numbers beyond 1000 for some children. Talk about the similar way of subtracting these numbers and subtracting numbers less than 100.

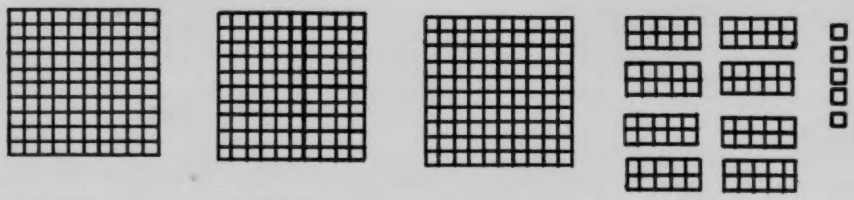
Pupil's book pages 123 - 125: Discuss the first page with the children. Use the other pages for independent work.

Finding the Difference Between Two Numbers

Wayne has 385 stamps. He put 152 of them in a stamp book. How many more does he have to put in the stamp book?

We write: 385 - 152 = ________

Think of 385 as: 300 + 80 + 5



We want to remove 152.

Think of 152 as 100 + 50 + 2

Think of removing 152 by ringing 1 set of one hundred, 5 sets of ten, and 2 sets of one.

Write the number of members in the set that is left.

___2___ hundreds, ___3___ tens, ___3___ ones.

We can write this: _200_ + _30_ + _3_ = _233_

$$\begin{array}{r} 300 + 80 + 5 \\ -\ (100 + 50 + 2) \\ \hline 200 + 30 + 3 \end{array} = \underline{\ 233\ }$$

Wayne has ___233___ more stamps to put in his book.

Computing the Difference Between Two Numbers

534 - 123 = 411

758 - 325 = 433

947 - 314 = 633

862 - 531 = 331

428 - 216 = 212

753 - 443 = 310

698 - 264 = 434

589 - 263 = 326

The Difference Between Two Numbers

Compute:

384 - 162 = 222

765 - 334 = 431

987 - 234 = 753

905 - 704 = 201

879 - 235 = 644

598 - 275 = 323

374 - 152 = 222

384 - 163 = 221

- Continue with the discussion

Return to the set of 185 stamps. Suggest that 78 stamps be removed from the set. Ask a child to write the expression (185 - 78). Then ask for ideas on finding the difference between 185 and 78, using both stamps and symbols. For example:

$$\begin{array}{rcrcr} 185 & & 100 + 80 + 5 & & 100 + 70 + 15 \\ \underline{-\ \ 78} & - & \underline{(70 + 8)} & - & \underline{(70 + \ 8} \\ & & & & 100 + \ 0 + 7 = 107 \end{array}$$

Children may suggest other sets of stamps to be removed. For each set suggested, ask if it will be necessary to regroup the sum and how they determine this. Emphasize that looking at the example carefully will help them to rename the sum correctly.

Pupil's book, pages 126 - 129: Discuss the first page with the children. The following page requires renaming the sum. Use the other pages for independent study. Whether the children use 1 or 2 steps in finding the differences between two numbers will depend on individual differences.

Renaming the Sum

1) 448 - 129
448 = 400 + 30 + 18

2) 572 - 227
572 = 500 + 60 + 12

3) 740 - 235
740 = 600 + 30 + 10

4) 571 - 329
571 = 500 + 60 + 11

5) 884 - 366
884 = 800 + 70 + 14

6) 793 - 458
793 = 700 + 80 + 13

7) 366 - 138
366 = 300 + 50 + 16

8) 857 - 248
857 = 800 + 40 + 17

Computing Differences

672
-235
437

591
-347
244

894
-488
406

750
- 237
513

Computing the Difference Between Two Numbers

348 - 129 Ans. 219

761 - 356 Ans. 405

532 - 318 Ans. 214

974 - 538 Ans. 436

883 - 647 Ans. 236

Finding Differences

Find the difference between each pair of numbers.

1) 391 and 269

Ans. 122

2) 994 and 267

Ans. 727

3) 792 and 269

Ans. 523

4) 545 and 237

Ans. 308

5) 434 and 329

Ans. 105

6) 289 and 168

Ans. 121

7) 678 and 339

Ans. 339

8) 387 and 178

Ans. 209

9) 963 and 238

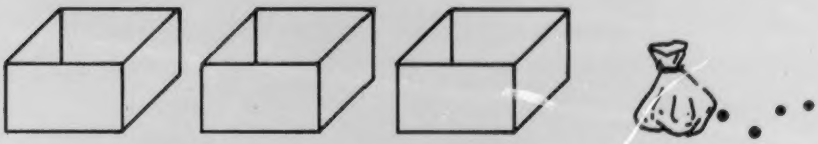
Ans. 725

10) 852 and 548

Ans. 304

- **Regrouping one hundred as tens**

Now we think of marbles. John has 314 marbles which he has counted and placed in bags and boxes. He has 3 boxes of 100 marbles and 1 bag of 10 and 4 single marbles.



He wants to give Bill 192 marbles.
How many marbles will he have left?

Write on the chalkboard:

$$314 - 192 = \underline{\quad}.$$

Ask another child to tell what will have to be done with the marbles. When the correct answer is given, ask another child to make a record of the procedure.

$$\begin{array}{r} 314 \\ \underline{-\ 192} \end{array} \qquad \begin{array}{r} 300 + 10 + 4 \\ \underline{-\ 100 + 90 + 2} \end{array} \qquad \begin{array}{r} 200 + 110 + 4 \\ \underline{-(100 + \ 90 + 2)} \\ 100 + \ 20 + 2 = 122 \end{array}$$

Continue the discussion by returning to the set of 314 marbles. Suppose John wanted to give Bill 180 marbles. Continue as suggested above.

Find differences for other pairs of numbers greater than 100. In every case ask the children if it is necessary to rename the sum in order to perform the computation. They should recognize that this is necessary if the tens number in the sum is less than the tens number in the known addend.

Select from the examples given below. Children should look at the example and if possible rename the sum in the most efficient form before beginning to work. In the expression, 418 - 276, an inspection of the tens number makes it apparent that the renaming of 418 should be 300 + 110 + 8 not 400 + 10 + 8. Of course, it will be necessary to permit children to work at their own level.

816 - 384 = ___
529 - 276 = ___
403 - 231 = ___

425 - 164 = ___
738 - 375 = ___
977 - 485 = ___

Pupil's book, pages 130 - 136: Use pages for independent work. Page 135 requires that the child look at the equations and ring either 100, 10 or 1 depending upon where the renaming is necessary. Sums which do not require renaming are included deliberately.

Renaming the Sum

1) 327 - 184
327 = 200 + 120 + 7

2) 809 - 582
809 = 700 + 100 + 9

3) 548 - 296
548 = 400 + 140 + 8

4) 739 - 546
739 = 600 + 130 + 9

5) 610 - 250
610 = 500 + 110 + 0

6) 768 - 473
768 = 600 + 160 + 8

7) 346 - 173
346 = 200 + 140 + 6

8) 218 - 192
218 = 100 + 110 + 8

Finding Differences

Find the difference between each pair of numbers.

1) 349 and 184

200 + 140 + 9
(100 + 80 + 4)
100 + 60 + 5 = 165

2) 901 and 290

Ans. 611

3) 847 and 283

Ans. 564

4) 638 and 293

Ans. 345

5) 427 and 295

Ans. 132

6) 539 and 284

Ans. 255

7) 504 and 242

Ans. 262

8) 928 and 296

Ans. 632

9) 588 and 297

Ans. 291

10) 650 and 180

Ans. 470

Computing the Difference

615 - 283

Ans. 332

719 - 237

Ans. 482

476 - 285

Ans. 191

827 - 265

Ans. 582

Computing the Difference

514 - 123

Ans. 391

947 - 254

Ans. 693

428 - 286

Ans. 142

618 - 264

Ans. 354

728 - 375

Ans. 353

Computing the Difference

871 - 390 Ans. 481
708 - 345 Ans. 363
557 - 273 Ans. 284
469 - 283 Ans. 186
673 - 280 Ans. 393

What must be renamed?

1)	347 - 128	100	[✓ 10]	1
2)	814 - 381	100	[✓ 10]	1
3)	73 - 48	100	10	[✓ 1]
4)	132 - 29	100	10	[✓ 1]
5)	49 - 27	100	10	1
6)	205 - 91	100	[✓ 10]	1
7)	981 - 257	100	10	[✓ 1]
8)	604 - 391	100	[✓ 10]	1
9)	876 - 59	100	10	[✓ 1]
10)	603 - 291	100	[✓ 10]	1
11)	540 - 239	100	10	[✓ 1]
12)	809 - 397	100	[✓ 10]	1

Some Problems to Solve

1. 969 children go to our school. There are 175 in the first grade. How many are not in the first grade?

969 - 175 = n

794 are not.

2. The third grade gave \$3.30 to the Red Cross. This was \$.50 more than the sixth grade collected. How much did the sixth grade give?

\$3.30 - \$.50 = n

They gave \$2.80.

3. The baseball team played 162 games. They lost 91 of them. How many did they win?

162 - 91 = n

They won 71 games.

4. Joe is reading a book. The book has 302 pages. He has read 150 pages. How many pages are left to read?

302 - 150 = n

He has 152 pages to read.

II - 7. Problem Solving

Objectives: To review techniques for solving story problems.
To develop further skill in solving problems involving comparison.
To develop skill in using one equation to help solve "two-step" story problems.

Vocabulary: (No new words.)

Materials:
4 wooden counting sticks;
7 pieces of 4" x 6" brown paper folded to represent buns;
8 pieces of yarn to shape into fish hooks on the flannel board;
5 felt fish;
other shapes for the flannel board.

Teaching Note:

Although children have had some experience in solving problems in previous sections, particular attention is given in this section to helping children develop strategies in solving problems. To minimize the computation, smaller numbers may be used to illustrate the idea. However, children may select much larger numbers in the teaching situation. The first part of the section develops as follows:

(1) Problem situations are considered without numbers whereby children may select whatever numbers they choose.

(2) Then we consider situations where not enough information is given.

(3) And, finally, we consider situations where too much information is given.

In the second part of this section, problems are considered whereby a comparison between two sets of objects must be made and "action" is taken on one or both of the sets of objects.

In the third part of this section, attention is given to the so-called two-step problems where two equations may be written or two ideas used in one equation.

Suggested Procedure:

Write the following story on the chalkboard:

> Mother had ______ pies. (1)
> She gave ______ pies to Grandmother.
> How many pies does Mother have now?

Read the story, using the word "some" for the blank space.

> What is unknown? What is it we are asked to find?
> (How many pies does Mother have after she gives some away?)
> What information is given? What does the story tell? (It tells that Mother had some pies and gave some of them away but it doesn't tell how many.)

Direct the children to read the story to themselves. Discuss the question asked and the information given. Relate the action to removing a subset of pies from a set of pies. Review with the children the fact that we may use equations to help solve the problem; however, the equation is not the answer to the story problem. For this problem we could use a subtraction equation. Ask the children to suggest numbers to be used in the blanks. Complete the story problem by writing the numerals and then direct the children to write the equation that will help them solve the problem on their paper. The equation should then be read so that all may check their work; e.g., $5 - 2 = n$. Be sure that the solution is stated, "Mother has _____ pies now."

- Too Little Information

Write the following story on the chalkboard:

Jimmy has a few marbles. (2)
Bill gave ____ marbles to Jimmy.
How many marbles does Jimmy have now?

Have the story read aloud. Discuss the question asked and the information given. The children should be quick to see that there is not enough information given to solve the problem. Encourage the children to tell what it is that they need to know. Erase the words "a few" and draw a blank there. Ask them to tell what kind of an equation may be used to help solve the problem. It will be an addition equation, since the set of marbles Bill gave to Jimmy was joined to Jimmey's set of marbles.

Ask the children to suggest numbers (e.g., 6 and 3) which could be used in the story. Fill in the blanks and then direct the children to write the equation on their own paper. Ask one child to write the equation on the chalkboard; e.g., $6 + 3 = n$. Direct the children's attention to the numbers named in the equation.

Bring out that each numeral names a number that is associated with a set of objects. The equation may be used to help find the answer but we must answer the question asked in the problem, "How many marbles does Jimmy have now?" The answer is, "Jimmy has _____ marbles."

- Too Much Information

Write the following problem on the chalkboard:

Sam needs 15 nails to build a box. (3)
He has 3 nails.
He has 5 screws
How many nails must he get?

Have the story read. Discuss the question asked and the information given. If no child comments about the screws, the children's attention should be directed to this statement by asking if all of the information given is needed to solve the problem. Relate the equation $15 = 3 + n$ to the problem.

Refer back to the problems on the board by numbering them 1, 2, and 3. Talk about whether knowing how to solve problem number 1 helped solve problem number 2.

Knowing how to begin by looking for the unknown and determining what information is important will help solve any problem. The same equation form cannot be used in solving both problems, though. The first one required a subtraction equation and the second one an addition equation. Bring out that problem number 3 was different from either of the others.

Pupil's book, page 137: Have the first story problem read and discuss which of the story problems on the chalkboard it is most like. Since it is most like the problem number 2 on the board, the children should be directed to write 2 in the small box in the upper left hand corner of the first story box.

Each child may think of the numbers he wishes to use in the story, the numerals should be written in the blanks, the equation written on the line below the story and the sentence completed.

The other three story problems should be completed independently in the same way.

Problem Solving

Jerry had ______ blocks.
He found ______ blocks.
How many blocks does
Jerry have now?

____

Jerry has ________ blocks.

Beth had ______ apples.
She gave ______ apples to Bill.
How many apples does
Beth have?

____

Beth has ________ apples.

Sue needs ______ bags.
She has ______ bags.
How many more bags does
she need?

____

Sue needs ______ bags.

Mother had ______ cookies.
Father took ______ of them.
How many cookies does
Mother have now?

____

Mother has ______ cookies.

Comparisons

Ask three boys to stand.

Tell the children that you want to make up a baseball team of nine members and you have only three members. How can you find out how many boys need to stand with the three boys already standing?

(You could keep picking boys until you have nine boys. You could think how many boys you would need, etc.)

Encourage the children to tell how many more boys are needed. Ask them to write an equation to show that the answer they gave was the correct response.

$$9 = 3 + 6$$

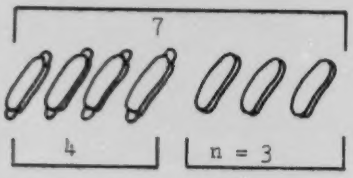
A child might write $9 - 3 = 6$, and justify it by saying that you know that you can find the missing addend by subtracting the known addend from the sum.

Give 4 counting stick "hot dogs" to one child and the 7 folded paper "buns" to another child.

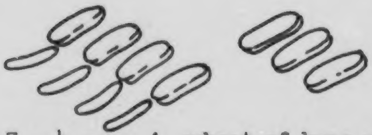
> Mary has 4 hot dogs and Judy has 7 buns. How many more buns are there than hot dogs?

Encourage the children to tell how they would pair the members of the two sets. If the hot dogs are paired with buns then the set of buns has two subsets, the subset with hot dogs and the subset without hot dogs. Have the equation written.

> $7 = 4 + n$. The set of buns was separated into two subsets, the subset with hot dogs and the subset without hot dogs.



If the buns are paired with the hot dogs, a subset of buns may be removed from the set of buns. Ask a child to write the equation to show this operation.

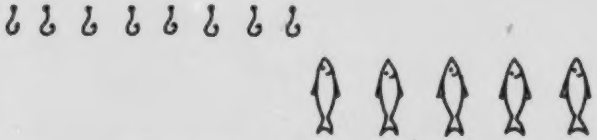


7 - 4 = n. A subset of buns was paired with the set of hot dogs. In this case you actually moved a set of 4 buns from the set of 7 buns. You can tell by the number of members in the subset of buns without hot dogs how many more buns than hot dogs.

Discuss the fact that other questions might have been asked about the hot dogs and buns. For example, how many more hot dogs do you need to fill the buns? How many fewer hot dogs do you have than buns?

Write the equation for each and have the question answered. (You need 3 more hot dogs to fill the buns. You have 3 fewer hot dogs than buns.)

- Place 8 yarn "fish hooks" on the flannel board. On the flannel board, but removed from the hooks, place 5 felt fish.



What questions might we ask about the fish and the fish hooks? (How many fish do we need to catch to have a fish for each hook? How many more hooks than fish? How many fewer fish than hooks? How many fish are needed to match the set of hooks?)
Is this problem similar to the last one? (Yes. Both problems are about comparing sets where one set has more members than the other set.)

Continue by discussing the first question given. Ask a child to write the equation that will help solve the problem. Name another child to show that the solution is true by moving the objects on the flannel board.

For example, if the equation were written $8 - 5 = n$, the child should pair the hooks with fish, moving the hooks to the fish, and thus removing from the set of 8 hooks a subset of 5 hooks to match the set of fish. If the equation were written $8 = 5 + n$, the child could pair the fish with the hooks and thus show hooks with fish and hooks without fish. The child might count 5 hooks and show that this subset is equivalent to the set of fish and the remaining subset uses up all the other members of the original set of hooks. The subset of hooks without fish will be equivalent to the set of fish needed in order to have a fish for each hook.

Ask the children to open their books to Pupil's book, page 138. Read the first story together. Direct the children's attention to the picture that is used to help solve the problem. Complete the page together.

Teaching Note:

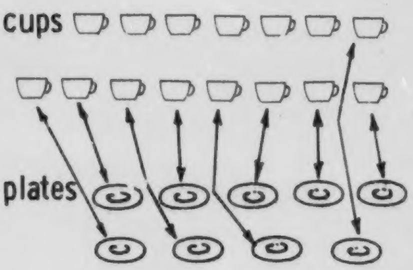
Some of the children may find this page very easy. These children may be encouraged to go directly to the pupil's pages which follow (139 - 141). Less able children may need to spend more time on page 138 under your direct guidance.

If the children are ready for more difficult problems involving comparison, Pupil's book, pages 142 - 144 may be used. You must make this decision for your own particular class situation. Again, very simple problems have been used in the beginning in order to be sure that the children understand the idea of comparison.

Problem Solving

Draw pictures to help solve the problem.

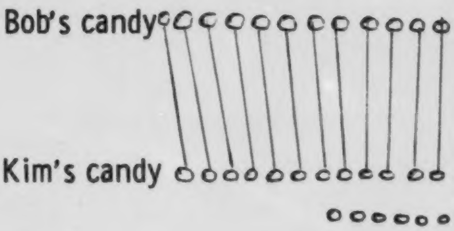
Judy and Susan were playing house.
Judy brought out 9 toy plates.
Susan brought out 15 toy cups.
How many more cups than plates did the girls have?



15 - 9 = n

There were 6 more cups than plates.

Bob and Kim went to the store to buy some candy.
Bob got 12 pieces of candy.
Kim got 18 pieces of candy.
Find how many more pieces of candy Kim had than Bob had.



18 - 12 = n

Kim had 6 more pieces of candy.

Solving Problems

1. Jan and Mark were going to play garage.

Jan had 12 toy trucks.

Mark had 21 toy cars.

How many more cars than trucks were there?

21 - 12 = n

There were 9 more cars than trucks.

2. Bill and Glenn were going to the store.

Bill had 33 cents.

Glenn had 18 cents.

How many fewer cents did Glenn have than Bill had?

33 - 18 = n

Glenn had 15 fewer cents than Bill had.

3. Susan's mother has 2 dozen pencils.

Susan has 9 pencils.

How many more pencils does Susan's mother have than Susan has?

24 - 9 = n

Susan's mother has 15 more pencils.

4. Jack ate 12 pancakes.

Father ate 9 pancakes.

Father ate how many fewer pancakes than Jack?

12 - 9 = n

Father ate 3 fewer pancakes.

5. Sally and Beth have 22 books.

Bob and Jim have 17 books.

How many more books do the girls have than have the boys?

22 - 17 = n

The girls have 5 more books.

6. Twenty-five crows were sitting on a fence.

Forty-one cows were in the field.

How many fewer crows than cows were there?

41 - 25 = n

There were 16 fewer crows than cows.

7. Tom caught 21 fish.
Father and Mother each caught 8 fish.
Find how many more fish Tom caught than
his parents caught.

21 - 8 = n

Tom caught 13 more fish than his parents caught.

8. There were 43 elm and 28 oak trees
in the park.
How many more elm trees than oak
trees were in the park?

43 - 28 = n

There were 15 more elm trees than oak trees.

Solving Problems

Find the answer and write the answer sentence.

1. Miss Brown had 78 sheets of red paper and 29 sheets of blue paper. Find how many fewer sheets of blue paper than red paper Miss Brown had.

   78 - 29 = 49

   Miss Brown had 49 more sheets of red paper than blue.

2. Miss Brown asked Judy to get the paint brushes. Judy got 32 wide brushes and 19 narrow brushes. How many more wide brushes than narrow brushes did she get?

   32 - 19 = 13

   Judy got 13 more wide brushes than narrow ones

3. The first box of colored chalk had 43 pieces. The second box of chalk had 28 pieces. How many more pieces were in the first box than in the second box?

   43 - 28 = 15

   There were 15 more pieces in the first box than in the second.

4. Miss Brown said that she had 63 pairs of scissors and that Miss Stone had only 38 pairs of scissors. How many fewer pairs of scissors did Miss Stone have than Miss Brown had?

63-38= 25

Miss Stone had 25 fewer pairs of scissors than Miss Brown

5. In the A parking lot there were 247 cars. In the B parking lot there were 173 cars. Find how many more cars were in the A lot than in the B lot.

247-173= 74

There were 74 more cars in lot A than in lot B.

6. There were 97 sport cars in the A lot. There were 129 standard cars in the A lot. How many fewer sport cars than standard cars were there in the A lot?

129-97= 32

There were 32 fewer sport cars than standard cars in lot A.

7. There were 67 sport cars in the B lot. There were 96 standard cars in the B lot. Find how many more standard cars than sport cars were in the B lot.

96-67= 29

There were 29 more standard cars than sport cars in lot B.

8. All together there were 150 station wagons in lots A and B. There were 31 trucks parked there. How many more station wagons than trucks were there in the lots?

150 - 31 = 119

There were 119 more station wagons than trucks in lots A and B.

- Two Problems in One

Write the following story on the chalkboard:

Farmer Brown had 5 pigs.
He bought 4 more pigs.
Then he sold 3 of his pigs.
How many pigs does he have now?

Ask a child to read the story. Discuss what is unknown. What information is given to help solve the problem?

Use manipulative materials to help the children visualize the 5 pigs he had and the 4 pigs he bought. Ask a child to show how he would solve the problem using the sets of objects. Direct the same child to show how he would write the equations that could be used to help solve the problem.

Encourage the other children to show how they might use a different method to solve the problem. Different equations should also be written.

For example: One child might join the two sets of pigs and then remove a set of 3 pigs. In this case, the equations would be written $5 + 4 = 9$, $9 - 3 = 6$. The answer to the question would then be given, "Farmer Brown has 6 pigs now."

Leave the story and the equations on the board. Read the following story to the children.

Father bought 3 bananas and 5 apples.
Mother bought 4 pears.
How many pieces of fruit did Mother and Father buy?

Write on the board:

3 bananas and 5 apples
4 pears

Discuss what is to be found and what information is given. Since the children have become familiar with the equation form using three addends, the equation will most likely be written $3 + 5 + 4 = 12$, and the answer given, "Mother and Father bought 12 pieces of fruit."

Through discussion bring out different ways of combining the sets of fruit.

> Maybe Father put the fruit he bought into one bag and Mother put the fruit she bought into another bag.
>
> If we are going to add the numbers in our equation in the same order that we joined the sets which numbers would we add first? $(3 + 5)$

Direct a child to write parentheses to show which numbers will be added first.

> $(3 + 5) + 4 = \underline{\quad}$
> $8 + 4 = 12$
>
> Let's pretend that the bananas were very ripe so the clerk put the bananas in one bag and the apples and pears in another bag. Where might the parentheses be written now?
>
> $3 + (5 + 4) = \underline{\quad}$
> $3 + 9 = \underline{12}$

It should be clear to the children that if you consider number only, it makes no difference which way you add. However, it sometimes helps to relate the equation closely to the problem and to write the parentheses in a certain place in the equation.

> What is the answer to the question asked? (Mother and Father bought 12 pieces of fruit.)

Direct the children's attention to the problem about Farmer Brown and his pigs. Begin the discussion by asking questions such as, "Why did we write two equations to help solve this problem? Are two equations necessary? Could you write one equation that would help you solve this problem?"

Encourage the children to try. If no one gives an answer, then you should write $(5 + 4) - 3 = n$. Have the children use manipulative materials to demonstrate the action. Continue by completing the computation and writing $9 - 3 = 6$.

Encourage the children to write the equation which would fit the problem when they think of removing the subset of pigs that the farmer sold from the set of pigs he had before joining to that set the set of pigs he bought. $(5 - 3) + 4 = n$, $2 + 4 = 6$.

Write the following story on the chalkboard:

> The mailman had 10 letters in his bag.
> He left 4 letters at Mrs. Smith's house.
> Then he picked up 3 letters that Mrs. Green wanted to mail.
> How many letters does the mailman have now?

Follow the same procedure with this problem. What is to be found? What do you know? Write the equation that will help solve the problem. $(10 - 4) + 3 = n$, and then writing $6 + 3 = 9$. Encourage the children to tell other ways of thinking about the problem. You might help by asking how many different letters the postman had. Encourage the children to try to write the equation that would help solve the problem in this way. If no child is willing to try, then you might write as you say:

> We can write $10 + 3$ because he had 10 letters and got 3 more from Mrs. Green. These parentheses will show that we add those numbers before we subtract any number. Now we subtract 4, the number of letters he left at Mrs. Smith's house. Our equation will read $(10 + 3) - 4 = n$. We will use this equation to find the number we will use

in our answer.

Have the computation completed and the question answered. (The mailman has 9 letters now.)

Pupil's book, page 145-146:

Read the first story problem and discuss whether it is similar to the problems they have just solved. Use the first problem as an illustration of the way in which the work is to be shown and completed.

Sets of Problems

Pupil's book, pages 147-154:

Three sets of problems follow. These pages may be used whenever it seems appropriate to provide opportunity for using mathematics in solving problems to continue the development of skill in problem solving.

Other problem sets will appear from time to time.

Problem Solving

Write the equation that will help solve the problem.

Put the ( ) where they belong in your equations.

1. Judy had 6 records

She bought 3 more records.

On the way home she broke 2 records.

How many records does Judy have now?

$n = (6+3) - 2$

$n = 9 - 2$

$n = 7$

2. Jim had 2 shirts and his mother bought 3 new shirts for him.

His grandmother sent a new shirt for his birthday.

Now how many shirts does Jim have?

$n = (2+3) + 1$

$n = 5 + 1$

$n = 6$

3. Beth borrowed 6 crayons from Susan.

That afternoon she returned 4 crayons to Susan.

Then she borrowed 3 crayons from Jerry.

How many borrowed crayons does Beth have?

$n = (6-4) + 3$

$n = 2 + 3$

$n = 5$

4. Mrs. White had only 4 eggs so she bought a dozen eggs.

How many eggs did she have after she put 6 eggs into a cake?

$n = (4+12) - 6$

$n = 16 - 6$

$n = 10$

5. 14 cars were in the parking lot.

6 cars came to park and 4 cars drove away.

How many cars were in the parking lot then?

$n = (14 + 6) - 4$

$n = 20 - 4$

$n = 16$

---

6. Mr. Black planted 4 oak trees.

Next he planted 3 maple trees.

Last of all he planted 5 elm trees.

How many trees did he plant?

$n = (4 + 3) + 5$

$n = 7 + 5$

$n = 12$

---

7. Mother made 8 red aprons and 5 blue aprons.

She gave 4 blue aprons away.

How many aprons does she have now?

$n = (8 + 5) - 4$

$n = 13 - 4$

$n = 9$

---

8. Sally had 12 cents.

She gave 5 cents to Bill.

Later Father gave 3 cents to Sally.

How many cents does Sally have now?

$n = (12 - 5) + 3$

$n = 7 + 3$

$n = 10$

Set 1

Solving Problems

Write an equation and complete the answer sentence.

1. The popcorn man had 75 bags of popcorn to sell.
At the end of the day he had 17 bags left.
How many were sold?

$75 - 17 = n$

58 bags of popcorn were sold.

2. Bill and Bob counted cars as they walked home.
Bill counted 67 cars and Bob counted 86 cars.
How many cars did they both count?

$67 + 86 = n$

They counted 153 cars.

3. In a spelling contest Jim's team made 32 points.
Henry's team made 17 points.
By how many points did Jim's team win?

$32 - 17 = 15$

Jim's team won by 15 points.

4. Sue picked flowers for her teacher.
She picked 49 daisies and a dozen tulips.
How many flowers did she pick?

$49 + 12 = n$

Sue picked 61 flowers for her teacher.

Set 2

Solving Problems

Write an equation and complete the answer sentence.

1. William has 14 pencils. If his mother gives him 12 more, how many pencils will he have?

   $14 + 12 = n$

   William will have 26 pencils.

2. James is 21 years old. He is 13 years older than his brother. How old is his brother?

   $21 - 13 = n$

   His brother is 8 years old.

3. John's teacher has 25 pieces of chalk. If she gives John 8 pieces, how many will she have?

   $25 - 8 = n$

   She will have 17 pieces of chalk.

4. If Pete spends 25¢ on oranges and 31¢ on bananas, how much will he have spent on fruit?

   25¢ + 31¢ = n

   He will have spent 56¢ on fruit.

5. The Carpenters' dog Rover just had 10 puppies. Their other dog, Fido, had 6 puppies a month ago. How many puppies did both dogs have?

$10 + 6 = n$

Both dogs had 16 puppies.

6. Mr. Barton is 40 years old. Mr. Hill is 19 years old. How much older than Mr. Hill is Mr. Barton?

$40 - 19 = n$

Mr. Barton is 21 years older than Mr. Hill.

7. If Mr. Jackson catches 14 fish and his wife catches 15 fish, how many fish do they catch in all?

$14 + 15 = n$

They catch 29 fish.

8. Mickey hit 54 home runs. He hit 20 more than Dave. How many home runs did Dave hit?

$54 - 20 = n$

Dave hit 34 home runs.

9. Tim had 13 pears. Jeff gave him 4 apples. How many pieces of fruit does Tim have now?

$13 + 4 = n$

Tim has 17 pieces of fruit.

10. A football club has 30 members. Only 14 members played in their big game. How many members did not play?

$30 - 14 = n$

16 members did not play.

11. Roger is 18 years old. He has a brother named Max. If the sum of Roger's and Max's ages is 32, how old is Max?

$32 - 18 = n$

Max is 14 years old.

12. Timothy needs 98¢. He has 25¢ now. How much will he have to earn before he has 98¢?

98¢ − 25¢ = n

He must earn 73¢.

13. Bill had 50 marbles. He gave Jerome 14 of them. How many marbles does Bill have now?

$50 - 14 = n$

Bill has 36 marbles.

14. Mr. Singer has 40 chickens. He bought a chicken house that can hold 90 chickens. How many more chickens will he need to fill his chicken house?

$90 - 40 = n$

He will need 50 chickens.

15. Patty had some jelly beans. Kim gave her 16 more and now she has 34. How many did she have at first?

$n + 16 = 34$ or $34 - 16 = 18$

She had 18 jelly beans.

16. There were 43 trees on one street. On another street there were 56 trees. How many trees were there on both streets?

$43 + 56 = n$

There were 99 trees on both streets.

Set 3

Solving Problems

Write an equation and an answer sentence.

1. Mary's sister was 15 years old. Mary was 5 years younger than her sister. How old was Mary?

$15 - 5 = n$

Mary was 10 years old

2. The boys in Mrs. Jones' class wanted to play baseball. They needed 18 members for two teams. There were only 11 boys on the field. How many more boys were needed before the game could begin?

$18 - 11 = n$

They needed 7 more boys

3. Carol had to walk 9 blocks to school. Jane had to walk 13 blocks. Which girl had to walk farther? How many more blocks did she have to walk?

$13 - 9 = n$

Jane had to walk 4 more blocks than Carol.

4. Alice's new baby sister weighed only 7 pounds. Alice weighed 35 pounds. How many more pounds did Alice weigh than her baby sister?

$35 - 7 = n$

Alice weighed 28 more pounds than her baby sister.

5. Susie baked 2 dozen cookies. She needed 3 dozen for the class party. How many more cookies did she have to bake?

$3 - 2 = n$ or $36 - 24 = x$

Susie had to bake one dozen more cookies or Susie had to bake 12 more cookies.

6. There were 34 children in the class. Nineteen of these were boys. How many girls were in the class?

$34 - 19 = n$

There were 15 girls in the class.

7. Sixty-two children had parts in a play. There were 80 parts to be filled. How many more children were needed?

80-62 = n

Eighteen more children were needed

8. George and Jerry rode their bikes 22 blocks from Jerry's house to the store. On the way home George stopped at his house which was only 7 blocks from the store. How many more blocks did Jerry have to ride to get home?

22 - 7 = n

Jerry had to ride 15 more blocks

9. Linda has earned 25 cents. She wants to buy a tea set that costs 59 cents. How much more money does she need?

59 - 25 = n

Linda needs 34 more cents.

★ II-8. Extensions

Objective: To provide additional material for more able pupils.

Vocabulary: (Indicated below)

Suggested Procedure:

The Pupil's book, pages 155-173, provide additional material on the **ideas developed** in this chapter. The items within each set increase in difficulty from the level of the regular lesson to a level that should challenge the brightest pupils. These pages are intended mainly for independent use by individual children. Very often one **idea** leads to the next, somewhat in the manner of **programed material**. You may want to point this out to the children who try these exercises.

Following are specific comments:

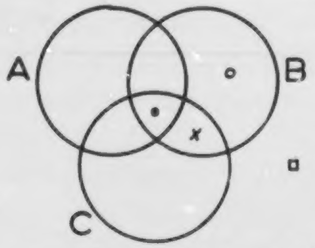
The set entitled Sequences and Sums gives practice in regrouping and reordering sums of several numbers.

The ideas in the set on Magic Squares may be quite difficult. Bright pupils may nonetheless find them interesting. The earlier ideas in this set are much easier.

Roman Numeral Arithmetic provides a nice example of a system with regrouping but without place value. This set may therefore be useful not only as a supplement but also as a step on the way to addition and subtraction of multidigit Arabic numerals.

★Overlapping Sets

1. Here are three circles A, B, C.



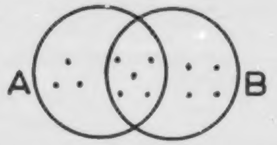
(a) Find a point that is inside all three circles. Mark that point with a dot.

(b) Now find a point that is inside circles B and C but outside circle A. Mark this point with a small X.

(c) Now find a point that is inside circle B but outside circles A and C. Mark this point with a small o.

(d) Now find a point that is outside all of the circles. Mark this point with a small box.

2. Here are two circles.



(a) Put five dots in the region that is inside both circles.

(b) Put three dots inside circle A but outside circle B.

(c) Put four dots inside circle B but outside circle A.

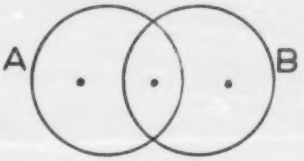
How many dots are in circle A? 8

How many dots are in circle B? 9

How many dots are in the picture all together? 12

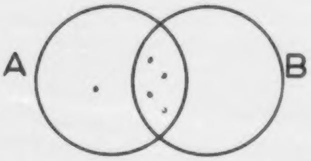
How many dots are both in circle A and in circle B? 5

3. Can you put 3 dots in this picture so that there are exactly 2 dots in circle A and 2 dots in circle B? yes



4. What is the smallest number of dots you can put in this picture and still have five dots in circle A and four dots in circle B?

5



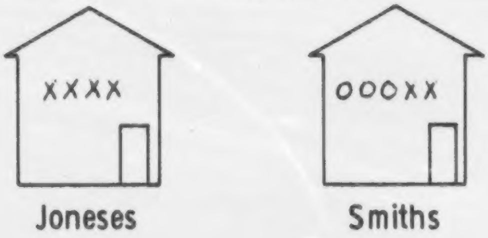
5. It is rainy today, so each pupil in Miss Black's class has brought either a raincoat or an umbrella. Six raincoats and seven umbrellas are hanging in the cloak-room. Two pupils brought both an umbrella and a raincoat. How many pupils are in Miss Black's class? 11

6. Mr. Adams has nine birds in his pet shop. Five of them are brightly colored and five of them have good singing voices. I would like to buy a brightly colored bird with a good singing voice. Do you think Mr. Adams has one? yes

Why? Four, at most, have poor voices. So one, at least, of the five brightly colored birds must have a good voice.

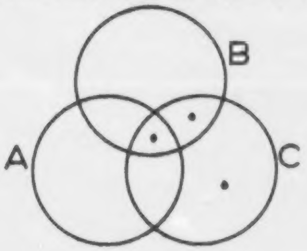
7. The Smiths and the Joneses are next door neighbors. The Smiths have 5 children, 3 of whom are girls. There are 6 boys in the two families. The Joneses have 4 children. How many of the Jones children are girls? ___0___

Here are the two houses. Put in X's for boys and O's for girls. This will help you find the answer.



8. Here are three circles A, B, and C. Can you put in three dots so that:

circle A will have one dot in it?
circle B will have two dots in it?
circle C will have three dots in it?



★ Sequences

1. 1 + 2 = __3__
   1 + 2 + 3 = __6__
   1 + 2 + 3 + 4 = __10__
   1 + 2 + 3 + 4 + 5 = __15__

2. 9 + 8 = __17__
   9 + 8 - 7 = __10__
   9 + 8 - 7 - 6 = __4__
   9 + 8 - 7 - 6 + 5 = __9__
   9 + 8 - 7 - 6 + 5 + 4 = __13__
   9 + 8 - 7 - 6 + 5 + 4 - 3 = __10__
   9 + 8 - 7 - 6 + 5 + 4 - 3 - 2 = __8__

3. 8 + 7 = __15__
   8 + 7 - 3 = __12__
   8 + 7 - 3 + 1 = __13__
   8 + 7 - 3 + 1 - 9 = __4__

4. 7 + 8 - 6 - 7 = __2__
   Let's change the order of the numbers:
   7 - 7 + 8 - 6 = __2__
   Is the answer the same? __yes_____

5. Let's try that again.

9 + 6 + 4 = 19

Now change the order of the numbers:

6 + 4 + 9 = 19

Is the answer the same? yes

Which order do you like better? the second

Why? 6+4 = 10 and 10+9 = 19 are easier than 9+6 = 15 and 15+4 = 19.

6. 7 + 9 + 3 + 1 = 20

Can you change the order of the numbers so that the addition is easier? yes

How? 7+3 +9 +1 = 10 + 10 = 20

7. 8 + 5 - 7 - 4 = 2

8. 3 + 9 + 7 - 1 = 18

9. 3 + 8 + 4 + 3 = 18

10. There are 2 planets closer to the sun than the earth. There are 6 planets farther from the sun than the earth. How many planets are there all together? (Do not forget the earth; it is a planet too!)

9

11. Last year Mr. Frank had these trees in his yard:

4 maples
5 oaks
7 elms
3 birches

During the winter a storm knocked down 2 birches and this summer the Dutch elm disease killed 4 of the elms.

How many trees does Mr. Frank have now? 13

12. Each day a jet airplane flies from New York to Chicago and then from Chicago to San Francisco. One day 30 passengers rode all the way from New York to San Francisco, 80 passengers rode only as far as Chicago, and 70 passengers got on at Chicago and rode to San Francisco.

How many people rode on the plane that day? 180

How many people were on the plane between New York and Chicago? 110

How many people were on the plane between Chicago and San Francisco? 100

★Sums

1. Here is a set of numbers:

3, ~~2,~~ 9, ~~6~~

Find a subset of these numbers whose sum is 8. Cross out the numbers you have chosen and write them into this equation.

__2__ + __6__ = 8.

The sum of the numbers left over should be 12. Write them in:

__3__ + __9__ = 12.

2. Do this one the same way. Cross out the numbers as you put them into the equations. Use each number only once.

~~1,~~ ~~7,~~ ~~5,~~ ~~8~~

__1__ + __5__ = 6

__7__ + __8__ = 15

3. Now do this one:

~~8~~ ~~4~~ ~~3~~ ~~9~~ ~~4~~

__8__ + __9__ = 17

__4__ + __3__ + __4__ = 11

Can you find a subset of three numbers that add up to 17, leaving a subset of two numbers that add up to 11?

__4__ + __9__ + __4__ = 17

__8__ + __3__ = 11

4. This time write in your own plus signs.

~~7,~~ ~~8,~~ ~~2,~~ ~~6,~~ ~~1~~

8 + 6 = 14

7 + 2 + 1 = 10

Find another way to do this one:

7 + 6 + 1 = 14

8 + 2 = 10

5. Now do these the same way:

(a) 5, 4, 8, 3

(8=8) 5+ 3 = 8

(5+4+3=12) 4 +8 = 12

(b) 9, 8, 3, 6

(8+3+6=17) 9 + 8 = 17

(9=9) 3 + 6 = 9

(c) 6, 5, 8, 4, 7

8 + 7 = 15

6 + 5 + 4 = 15

(d) 9, 8, 7, 6, 2

(8+7+2=17) 9 + 8 = 17

(9+6=15) 7+6 + 2 = 15

6. Look back at Problem 5. How many ways can you find to do each of those examples.

(a) 2

(b) 2

(c) 1

(d) 3

7. Here are some with three equations to fill in. Remember to use each number only once.

(a) 7, 2, 9, 3, 6, 6

3 + 6 = 9

2 + 9 = 11

7 + 6 = 13

(b) 13, 5, 9, 3, 2, 9

9 + 9 = 18

13 + 2 = 15

5 + 3 = 8

(c) 8, 6, 9, 8, 5, 9

9 + 5 = 14

6 + 9 = 15

8 + 8 = 16

8. Make two equations out of these numbers. Use each number once and only once. If you like you may put two or more numbers on the right side of the equation.

1, 2, 3, 4, 5, 6, 7

3 + 4 = 2 + 5

1 + 6 = 7

9. This time make three equations. Remember to use each number once and only once.

5, 13, 7, 5, 9, 16, 13

9 + 7 = 16

5 = 5

13 = 13

★Magic Squares

1. Here is an array of numbers.

1	5	2	$\boxed{8}$
3	1	8	$\boxed{12}$
4	2	2	$\boxed{8}$
⑧	⑧	⑫	

Add the numbers in the rows and put the sums you get in the boxes at the right. The first one is done for you. Now add the numbers in the columns and put the sums in the circles along the bottom.

What is the sum of the numbers in the boxes? 28

What is the sum of the numbers in the circles? 28

Now look back at the array.

What is the sum of the nine numbers in the array? 28

Are the three sums you have just found all the same? yes

Why? A set of numbers to be added can be put in any order or grouped in any way.

2. Here is an array with some numbers missing. Fill in the missing numbers so that the row sums and the column sums are all correct.

0	0	1	1
0	1	1	2
1	1	1	3
1	2	3	

3. Now try this one:

7	7	6	20
7	8	5	20
6	5	9	20
20	20	20	

4. How many ways are there to do this one? 16

(You can put any number from 0 through 15 in a blank box. After that you have no choice.)

9	0	15	24
15	9	0	24
0	15	9	24
24	24	24	

5. Here is one with four rows and four columns.

2	4	6	8	20
7	6	7	0	20
8	3	0	9	20
3	7	7	3	20
20	20	20	20	

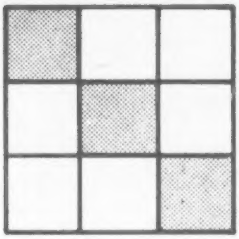
6. To do this one use each of the numbers

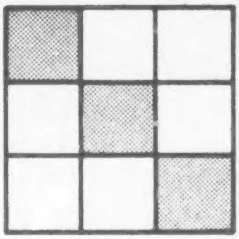
1, 2, 3, 4, 5, 6, 7, 8, 9

once and only once.

2	9	4	15
7	5	3	15
6	1	8	15
15	15	15	

7. Two subsets of an array are called diagonal subsets. In the arrays below the diagonal subsets are shaded:





Now do Problem 5 in such a way that the sums of the diagonal subsets are also to equal to 15.
The array you will find is called a "magic square."

★Roman Numeral Arithmetic

In this lesson we are going to learn to do some arithmetic with Roman numerals. You have probably seen Roman numerals on clocks or in books.

Here are the first twelve:

		5	V	10	X
1	I	6	VI	11	XI
2	II	7	VII	12	XII
3	III	8	VIII		
4	IIII	9	VIIII		

This is the way the numerals were written in the early days of Rome. In later times 4 was sometimes written IV and 9 was sometimes written IX. In this lesson, however, we will write 4 with four I's and 9 with a V and four I's in the manner of the early Romans. This will make the arithmetic easier.

Let's begin by writing some more Roman numerals. The Romans used these letters:

I for 1
V for 5
X for 10
L for 50
C for 100

They also had some more letters for greater numbers, but we won't talk about those now. To find out what number a Roman numeral stands for, you just add all the numbers that the letters stand for. For example:

XI = 10 + 1 = 11

Here are some other examples:

XVI = 10 + 5 + 1 = 16

XIII = 10 + 1 + 1 + 1 = 13

CLXXV = 100 + 50 + 10 + 10 + 5 = 175

1. What numbers do these Roman numerals stand for? Write out the sum as shown above.

VIII = 5+1+1+1 = 8

LXV = 50 + 10 + 5 = 65

CXI = 100 + 10 + 1 = 111

XXXVI = 10+10+10+5+1 = 36

CCLXII = 100+100+50+10+1+1 = 262

2. Here is a simple addition in Roman numerals:

VI + II = VIII

To add VI and II all you have to do is put together all the letters in both numerals. Think about why this is so. Here are some more additions that can be done in this simple way:

XI + I = XII

XXV + II = XXVII

X + XVI + III = XXVIIII

LX + XV = LXXV

Now check your work by changing the Roman numerals into your everyday numbers.

3. The early Romans always wrote the letters in order: first the C's, then the L's, then the X's, then the V's, then the I's. Sometimes to do addition you have to rearrange the letters. Try these. The first one is done for you.

XII + VI = XVIII

XII + V = XVII

XXIII + VI = XXVIIII

LXI + VII = LXVIII

CXV + LI + X = CLXXVI

4. Now try these. The first one is done for you.

CXI + LVI = CLXVII

CXXII + LVI = CLXXVIII

V + CCXIII + LI = CCLXVIIII

LII + XXXI + CV = CLXXXVIII

5. So far we have just put together all the letters in the numbers to be added. Sometimes addition is a little more complicated. If, for example, we just put together the letters in this addition example:

III + II

we get IIIII. But the Romans never wrote five I's together. Instead, they wrote V. Here are the rules the Romans used:

(1) No numeral ever has more than four I's in it.
(2) No numeral ever has more than one V in it.
(3) No numeral ever has more than four X's in it.
(4) No numeral ever has more than one L in it.
(5) No numeral ever has more than four C's in it.

6. The Romans also used these letters:

D for 500
M for 1000

What do you think the Romans' rule for D was?

(6) No numeral ever has more than one D in it.

7. Now use these rules when you do the following additon examples. The first two are done for you.

III + II = V
XIII + III = XVI
XXXIIII + III = XXXVII
CII + XI + LII = CLXV

CXII + XXII + I = CXXXV

DX + DI = MXI

8. Now do these:

V + V = X

CV + XVI + V = CXXVI

VI + XV = XXI

LV + XV + VII = LXXVII

MCXI + V + XVII = MCXXXIII

VIII + III = XI

9. Now do these:

XX + XXXV = LV

LV + CLV = CCX

XXII + XXXIII = LV

MLV + XV + LVII = MCXXVII

CLVI + LXII + LII = CCLXX

XXX + XV + DVI = DLI

10. Our numerals 1, 2, 3, 4, ... are called Arabic numerals. Write these problems in Roman numerals. Then do them in Roman numerals. Check your answer by adding the regular way. The first one is done for you.

27 + 18 = 45

XXVII + XVIII = XXXXV

55 + 24 = 79

LV + XXIIII = LXXVIIII

63 + 14 = 77

LXIII + XIIII = LXXVII

107 + 86 = 193

CVII + LXXXVI = CLXXXXIII

36 + 335 = 371

XXXVI + CCCXXXV = CCCLXXI

1247 + 115 = 1362

MCCXXXXVII + CXV = MCCCLXII

11. Now try some subtraction. Figure out the rules for yourself.

XXIII - XII = XI

VIIII - III = VI

XVII - XVI = I

X - V = V

X - VII = III

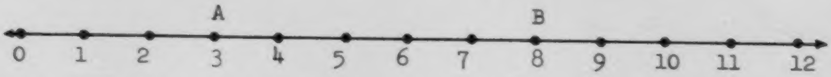
L - XX = XXX

Chapter III

DESCRIBING POINTS AS NUMBERS

Background

Some of the most important insights in mathematics are those arising from the interplay of geometry and arithmetic. Already we have often used a number line as a convenient way of representing, geometrically, relations among numbers. For example, the number relation $8 > 3$ is represented geometrically in the figure below by the fact that point B lies to the right of point A on $\overleftrightarrow{AB}$. (The symbol $\overleftrightarrow{AB}$ means "the line passing through (or lying on) the two distinct points A and B". Thus $\overleftrightarrow{AB}$ and $\overleftrightarrow{BA}$ have the same meaning.)

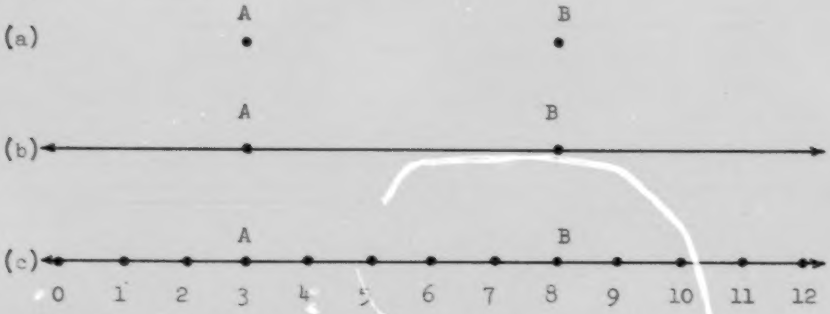


In this chapter we assume a different attitude. That is, we think of numbers as means of identifying points. Thus, in the figure above we identify or describe point A by the number 3 and point B by the number 8. Then we use these numbers (not the points which they identify) to obtain geometric facts. For example, the length of $\overline{AB}$ above is 5 units. The 5 is computed as $8 - 3$ from the numbers describing A and B. (The symbol $\overline{AB}$ denotes the line segment having points A and B as endpoints. Thus $\overline{AB}$ and $\overline{BA}$ have the same meaning. Therefore the length of $\overline{AB}$ is the same as the length of $\overline{BA}$.) Notice that there are no arrowheads in the symbol for a line segment.

A number used to describe a point is called a coordinate of that point. If the mathematical word coordinate seems difficult for children, you may use the phrase "number describing point A" as long as it seems necessary to do so.

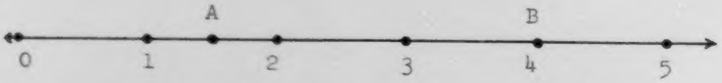
In this chapter we use only whole numbers as coordinates. Thus many points of the line are not assigned coordinates. For example, no point in the figure between that with coordinate 0 and that with coordinate 1 has been assigned a coordinate. Also, no point to the left of the 0-point has a coordinate. The pupil is invited to notice these omissions as hints that later new numbers may be invented and assigned to these points.

The several parts of the figure below may help us appreciate how free we are in deciding how coordinates may be assigned to points. We assume in (a) that the two points A and B are given. They determine uniquely a line $\overleftrightarrow{AB}$ which we represent in (b). On this line we choose, completely arbitrarily (and thus in any specific case, completely according to *whim* or convenience), a point which we identify by the number 0.



Also in (c) we have *arbitrarily* chosen both the *unit of length* (for example, distance between 0-point and the point which we will identify by the number 1.), as well as the *direction* from 0-point to 1-point. Once we have done all *three* of these things (and each may be done independently of *both* the others), the points which will have coordinates respectively 2, 3, 4, and so on, are all uniquely determined. For example, in (d) below, the choice of 0-point and the choice of direction are the same as in (c), but the choice of unit distance is different (in this case, the unit in (d) is twice that in (c)).

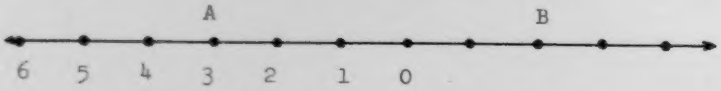
(d)



The misfortune here is that in this case the point A does not have a whole number coordinate. If we wanted to discuss A and B in terms of whole number coordinates we would not make these three choices of 0-point, direction, and unit.

In (e) we have chosen a different 0-point, the opposite direction, and the same unit as in (c).

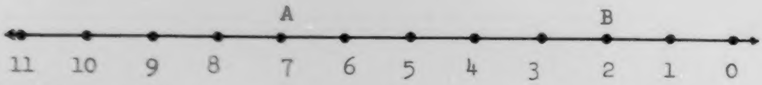
(e)



This set of choices also is unfortunate, since in this case point B does not have a whole number coordinate.

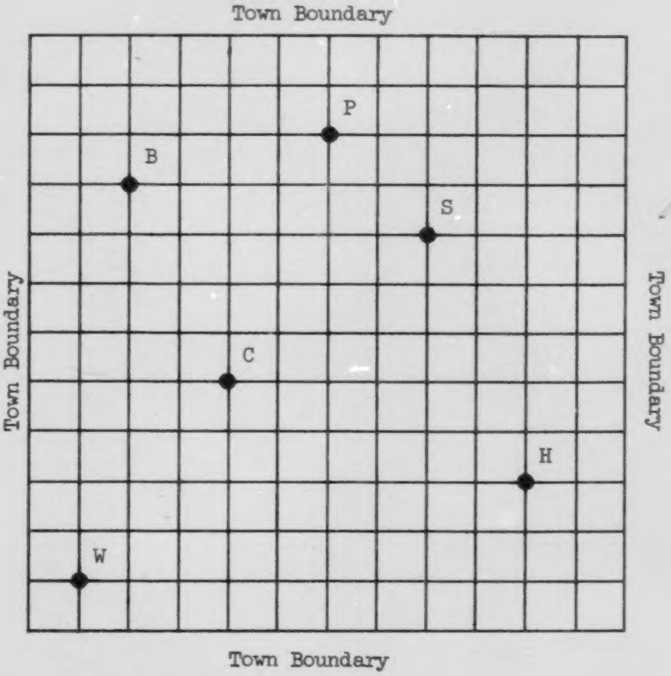
In (f) we have chosen the same unit and direction as in (e), but a different 0-point. It is clear again that the coordinates yield the information that the length of $\overline{AB}$ is 5 units.

(f)



These examples are intended to make it clear that there is a great deal of freedom in assigning coordinates to points. In order to simplify later work, let us agree that if A and B lie on a horizontal line, we shall take the direction from the 0-point to the 1-point as from left to right as we face the line.

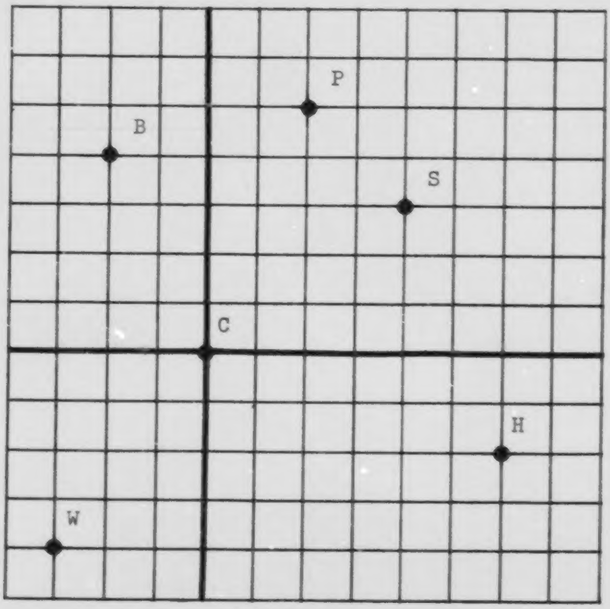
Sometimes we wish to assign coordinates to points in a plane. The idea may be illustrated in the following situation. Imagine that a town is laid out in square blocks with one set of streets running east and west, and the other set running north and south as in the figure below.



The waterworks is indicated by point W, the school by S, the civic center by point C, the ballpark by B and the hospital by H. The problem is to use whole numbers to describe the location of the arbitrary point P in the town. We simplify the problem (and make its solution possible) by considering only those points P at the intersections of streets (corners of squares).

If two points among W, C, S, B, and H lay on the same street, we could use that street to get started. They do not. So we pick some point arbitrarily. To get started, we pick C. Then, for example, S is 4 blocks east and 3 blocks north of C. If we were at C when someone asked for directions to the hospital (H) we might say, "Go six blocks east and two blocks south," or "Go two blocks south and six blocks east." If we want the bare, unadorned numbers to indicate the location of S, we might use an ordered pair of numbers. We might agree that the first number is to be used to indicate location along a left-right (or west-east) direction and the second (right hand) is to indicate location along a bottom-top (or south-north) direction. With this understanding, and with C as the starting point, we would indicate the point S with the pair of numbers, 4 and 3, written as (4, 3).

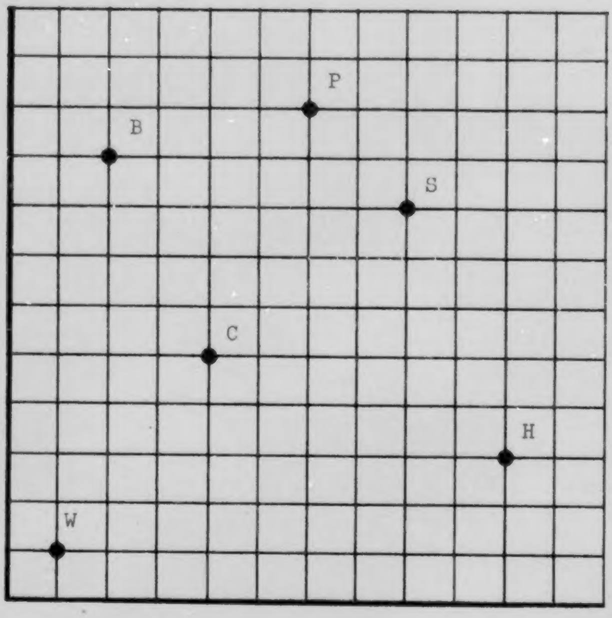
We now have two coordinates for a point instead of one coordinate which we found sufficient when we were working with points on a line. We can connect our first coordinate (4) with a "first" number line and our second (3) with a "second" number line in the following manner. The figure below shows the diagram of the town with a west-east line and a south-north line drawn in, each passing through C. On each of these the coordinate of **C** is 0, so that the number pair indicating C is (0, 0).



The chosen direction along the west-east line is to the right (eastward) and the chosen direction along the south-north line is upward (northward). The chosen unit on both lines is that representing one block. We see easily that the coordinates of point P are (2, 5) (why not (5, 2)?). Those of S are (4, 3) (why not (3, 4)?).

Similarly, we can give the coordinates of any point (intersection) which is to the right of the heavy vertical line and also above the heavy horizontal line. On the other hand, if we have given any number pair in which the first number is one of 0, 1, 2, 3, 4, 5, 6, 7 and 8 and the second number is one of 0, 1, 2, 3, 4, 5, 6, and 7, we can locate the indicated point of intersection. Thus the northeast corner of the town has the coordinates (8, 7); it is 8 blocks east and 7 blocks north of the civic center.

There is one drawback in our arrangement. Some of the points (intersections) in the town have no whole number coordinates. In particular, we cannot give coordinates for the waterworks, the ballpark, nor the hospital. To overcome this difficulty, we shall later simply use integers, and then W, B, and H will have coordinates (-3, -4), (-2, 4) and (6, -2) respectively. For the present, we accomplish this by choosing another starting point which will be the 0-point on each of two new lines.



With the new starting-point (origin) and the new lines drawn as shown, it is easy to see that the named points (intersections) and their respective coordinates are:

W	(1, 1)
C	(4, 5)
B	(2, 9)
P	(6, 10)
S	(8, 8)
H	(10, 3)

Hence, all the named points now have coordinates. Furthermore, all the points (intersections) in town now have coordinates.

A more important feature to notice is that the relations among the points did not change when we changed their coordinates. Thus, the school (S) is still 4 (i.e., 8 - 4, 8 being the new first coordinate of S and 4 being the new first coordinate of C) blocks east and 3 (i.e., 8 - 5) blocks north of the civic center (C).

It should be noticed that this last coordinate system is the only one which assigns whole number coordinates to all the points (intersections) in this town.

As an application of coordinates, an examination is made of the effects on a figure of performing certain operations on the coordinates. For example, Pupil Page 191 shows a triangle ABC determined by A(2, 6), B(6, 1), C(3, 10). Three new points P, Q, R, are obtained by adding 7 to each first coordinate and 4 to each second coordinate. The new triangle PQR is congruent to the original one. This process of obtaining one figure from another by adding numbers to the coordinates is called a translation. Similarly, on page 201 in the pupil's book is a triangle ABC. When all coordinates are multiplied by 2, we obtain a second triangle STW which is similar to ABC. That is, the angles of the old and new triangles are congruent, but the measures of the lengths of the sides have all been multiplied by the same number, in this case 2. This provides a method for enlarging a figure.

No attempt is made to explore systematically the properties of similar figures, but examples are given of a few scale drawings with application to the use of the scale of miles on a map.

The first four sections of the chapter do not involve any concepts the pupils have not had. The last three, however, involve ideas of multiplication, specifically that of multiplying all coordinates by some number. You may find it desirable to postpone the study of these last three sections until more has been done with multiplication.

You should think of this chapter as "opening a door" to several ideas that will grow in meaning for the child as he continues his education. Full mastery is not expected.

III-1. Coordinates of Points on a Line

Objective: To introduce and use the idea that a point on a line may be described by a number.

Vocabulary: Coordinate.

Materials: Number line on chalkboard.

Suggested Procedure:

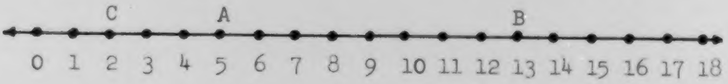
(In readiness for this lesson draw a number line on the chalkboard.)

Ask the pupils if they have ever ridden along a road on which there was a little marker each mile to tell the distance from some point. (This is quite common on dual lane roads and on very old roads in some parts of the country.) Ask them to imagine that they are riding on such a road when suddenly the car runs out of gasoline. It stops beside one of the mileage markers on which is the numeral 17. Fortunately, there is a phone nearby so the driver goes to phone the service station to send a truck with gasoline. Raise the question as to how the driver, who has never been on the road before, can tell the man at the service station where to bring the gasoline.

From the discussion, elicit the idea that the driver can ask that the truck be sent to the mileage post marked 17. Explain that what the driver did was to use a number, in this case 17, to describe the point where the car was to be found. Continue the discussion to bring out the fact that the road with its mileage markers is essentially a number line, except that it probably is not straight.

A number used to describe a point is called a coordinate of the point.

Turn now to a number line you have already drawn on the board.



Notice that on such a line there are many points which can be described by numbers, just as the driver described the location of the gasless car. For example, point A is described by the number 5, point B is described by the number 13, and point C is described by the number 2. Review the idea that the segments between consecutive marked points are all congruent to one another, and that any of these segments or any segment congruent to them may be called a unit segment for this number line.

It should be brought out that we have not described all points on the line by numbers. Such questions as the following may help.

> Does every whole number describe a point?
> (Yes.)
>
> Is every whole number a coordinate of a point?
> (Yes.)

The two questions above are the same.

> Is every point described by a whole number?
> (No.)
>
> Does every point have a whole number as a coordinate? (No.)

The two questions above are the same.

> Show a point which does not have a whole number coordinate.

Bring out that there are at least two kinds of points which have not been assigned whole number coordinates. First, there are the points between marked points; for example, there is the point midway between the points marked 0 and 1. Some child probably will be able to suggest that we already have a number to use as a coordinate for this point, namely, $\frac{1}{2}$. You should agree with this and indicate that if we wish to use rational numbers we can assign coordinates to many other (but not all other) points on the line. If we wish to use only whole numbers, however, we can only indicate that these other points are between the marked points. The above-mentioned midway point is between the point whose coordinate is 0 and the point whose coordinate is 1. Notice that if the car had run out of gasoline a few seconds later, the driver could have told the service man only that the car was stopped between the 17-mile marker and the 18-mile marker (assuming that he was going away from the 0 marker.)

The other points to which no coordinates have been assigned are those to the left of the starting point, or origin, whose coordinate is 0. Probably some child will suggest using negative numbers as coordinates for these points. If so, you should agree that eventually we will use negative numbers to describe these points.

Having given this "inkling" of things to come, indicate that for now we will use only whole numbers as coordinates of points.

Ask the pupils to consider segment $\overline{BC}$ of the number line on the board. The endpoints of this segment are B and C. In our case, B is the right-hand endpoint and C is the left-hand endpoint.

> If B is to the right of C, what is the relation between the coordinates of B and C ? (The coordinate of B will be greater.)
>
> What is the length of $\overline{BC}$? (The length of $\overline{BC}$ is 11 units.)

If there is any question, indicate that we mean length using the unit segment on the line. By discussion, elicit the idea that there are several ways to find the length of $\overline{BC}$. One way would be just to count the segments. Lead to the idea, however, that you can always get the length of a segment by subtracting the coordinate of the left-hand point from the coordinate of the right-hand point. Thus, the length of $\overline{BC}$ is $(13-2)$ units or 11 units. Similarly, the length of $\overline{AB}$ is $(13-5)$ units or 8 units. Also, the length of $\overline{AC}$ is $(5-2)$ units or 3 units.

Try a few examples of the following type, having the points marked clearly on the line.

The coordinates of P, Q, R, and S are 6, 1, 10, and 8, respectively.

Find the lengths $\overline{PR}$, $\overline{RQ}$, and $\overline{QS}$.

If one moves from P to R, then from R to Q, and finally from Q to S, how far has he moved in all? (20 units.)

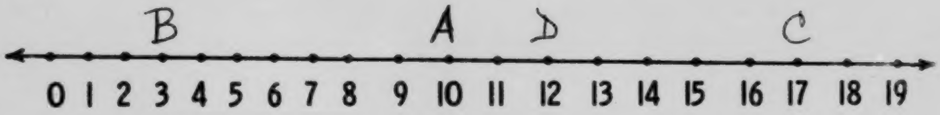
At the end of all this moving, how far is he from his starting point P? (2 units.)

Pupil's book, pages 175 - 176:

This page provides opportunity for children to name coordinates of points and then to find lengths of line segments.

Describing Points by Numbers

1.



Mark points A, B, C, D.

Point A has the coordinate 10.

Point B has the coordinate 3.

Point C has the coordinate 17.

Point D has the coordinate 12.

Complete the following:

The length of $\overline{AB}$ is (10 - 3) units or 7 units.

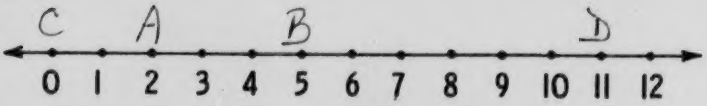
The length of $\overline{BC}$ is (17 − 3) units or 14 units.

The length of $\overline{CD}$ is (17-12) units or 5 units.

The total number of units in $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ is 26 .

The distance from A to D is 2 units.

2.



Mark points A, B, C, D.

Point A has the coordinate 2.

Point B is 3 units to the right of A.

Point C is 5 units to the left of B.

Point D is 11 units to the right of C.

B has the coordinate 5.

C has the coordinate 0.

D has the coordinate 11.

The length of $\overline{BC}$ is 5 units.

The length of $\overline{AD}$ is 9 units.

III-2. Motions on the Number Line

Objectives: To develop the understanding that the coordinate of a point depends on the choice of zero point, unit distance, and direction.

To consider motions on the number line.

Vocabulary: (No new words.)

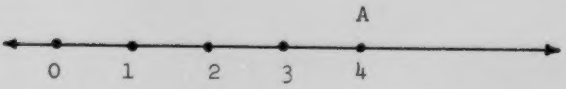
Materials: Colored chalk.

Suggested Procedure:

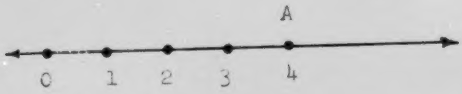
It is important that children understand that describing a point by a number as we have done implies certain choices, specifically a choice of a zero point, a unit distance, and a direction. Some such activities as the following should bring these ideas out explicitly.

Draw a line on the chalkboard and mark a point A on it as shown.

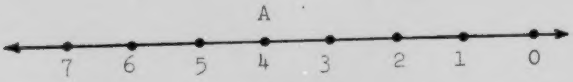
Ask if the coordinate of point A is 4 (or if A is described by the number 4). The answer presumably will be that you can't tell. Ask why you can't tell. This should bring out the idea that to say A has coordinate 4 would mean it is four units (in a given direction) from the zero point and that here there is no indication of either the zero point or the unit of length. Have a pupil come to the board and show a choice of unit length and then the zero point so A will have coordinate 4. The figure may look somewhat as follows:



Now ask another pupil to show a different choice in which A has coordinate 4. (This may be better done on a different drawing.) It might look as follows.



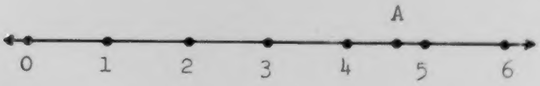
If no pupil has suggested using the opposite direction to show a different choice; suggest the following:



It may be desirable specifically to discuss these selections as follows.

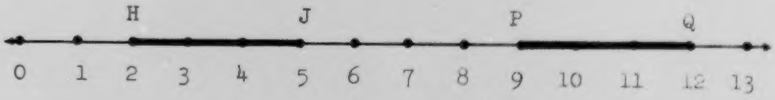
> It is natural for us to think of using the left-to-right direction on our number line because we read and write our language that way. However, there is nothing, except possibly convenience, to prevent our using a right-to-left direction. The resulting descriptions of points by numbers as well as the results we get based on them, would be just as good, just as accurate, and just as easy to read and to work with as those we get by using a left-to-right direction. In fact, mathematicians, scientists and engineers do not hesitate to use a right-to-left direction in those cases in which such use makes their work more convenient. However, we shall use the left-to-right direction exclusively in order to avoid confusion.

Now have a third pupil indicate choices so A would have a different coordinate (or be described by a different number), say 6. Finally, have some child show a choice in which A is not described by any whole number. Such a drawing might be like this.



Observe that sometimes, when we use coordinates (numbers to describe points), we find that the zero point and distance have already been chosen. This was illustrated in the last lesson by the automobile driver who ran out of gasoline. For him the numbers had already been assigned to the points. Sometimes we make the choice ourselves.

It would be well to set the following discussion in a story framework. The following is one possibility. Indicate that Henry and John are playing a game. The game represents an automobile race. They take turns in spinning a dial that tells them how many spaces to move. They have each played once. Henry spun a 2 and so placed his little car at H which is the point described by the number 2. John then spun a 5 and so placed his little car at J. Have the information entered on a number line drawn on the board as you proceed. Have $\overline{HJ}$ drawn in colored chalk so it will stand out.



It is Henry's turn again and he spins a 7, so he moves his little car 7 spaces to the right. Call this point P. Have the coordinate identified and have P marked on the line. Then John plays again. He also spins a 7 and moves to point Q which should be identified and marked on the line.

Have $\overline{PQ}$ drawn with chalk of a different color.

Discuss with the pupils the relation of $\overline{HJ}$ and $\overline{PQ}$. Such questions as the following may be suggestive.

> Who was ahead after they had played once? (John.)
>
> How much ahead was he? (3 units.)
>
> How did you find this? (5 - 2.)
>
> Is this the length of $\overline{HJ}$? (Yes.)
>
> Who was ahead after they had each played twice? (John.)
>
> How much ahead was he? (3 units.)
>
> How did you find this? (12-9.)
>
> Is this the length of $\overline{PQ}$? (Yes.)
>
> What relation is there between $\overline{HJ}$ and $\overline{PQ}$? (They are congruent. They have the same length.)

Discuss with the children what the result would have been if Henry and John had each spun a 4. Would the new segment $\overline{PQ}$ this time be congruent to $\overline{HJ}$? Why? Suppose they had each spun a 9 ?

We want the pupils to realize that as long as Henry and John advance by the same amount, then John's lead over Henry will get neither larger nor smaller. That is, the segments $\overline{HJ}$ and $\overline{PQ}$ will be congruent.

To see this congruence another way, make a model of $\overline{HJ}$ by marking it off with dots on the edge of a paper. Then slide this paper along the line, noticing that the two dots slide the same distance. Thus, when the left hand dot slides 7 units from H to P, the right hand dot slides 7 units from J to Q and $\overline{HJ}$ is congruent to $\overline{PQ}$.

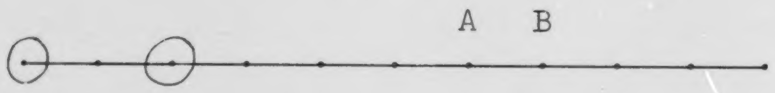
Lead to the general understanding that if the same number is added to the coordinates of the endpoints of a segment, the new segment is congruent to the old one.

Pupil's book, pages 177 and 180:

These pages may be used either independently or working together.

Motion on a Line

1. When George goes home from school he passes a long fence. The picture shows the fence. The dots are the fence posts.

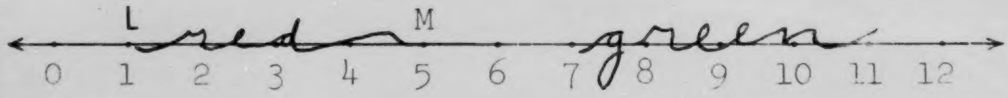


George likes to describe the posts with whole numbers.
He describes post A by the number 4.
He describes post B by the number 5.

Draw a ring around the post he describes with the number 0.
Can George describe all the posts with whole numbers? No

Sally does not like the way George describes the posts.
She says the numbers that describe A and B should be 6 and 7.
Put a cross on the post Sally describes by the number 0.
Can Sally describe all the posts by whole numbers? Yes

2. Pretend this number line shows a railroad track.



A train is on the track.

Its ends are at L and M.

Color red the track where the train is standing.

Point L is described by the number 1.

Point M is described by the number 5.

The length of the train is 4 units.

The train moves 6 units to the right.

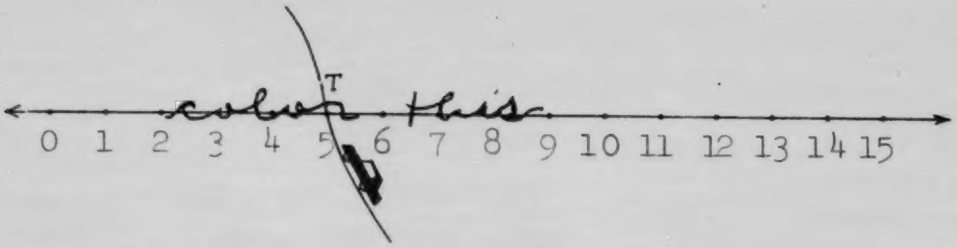
Call the new endpoints P and Q.

Point P is described by number 7.

Point Q is described by number 11.

Color green the track where the train is now.

3. Pretend the number line is a railroad track.



A train is on the track.

Its ends are described by numbers 2 and 9.

Color the track where the train is standing.

A road crosses the track at T.

A car is on the road.

Can the car cross the track? No

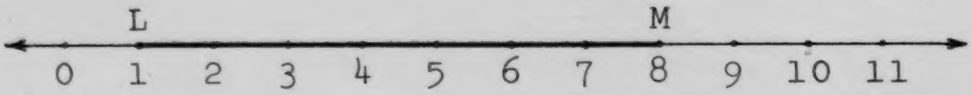
The train moves to the right.

The front of the train is described by the number 13.

The back of the train is described by the number 6.

Can the car now cross the track? Yes

4. Pretend this number line shows a railroad track.



A train is on the track.

Its ends are at L and M.

A road crosses the track at a point X.

Point X is described by the number 51.

Can you imagine the point X? Either Yes or No.

The train moves 46 units to the right and stops.

Its ends are described by the numbers 47 and 54.

Is the train across the road? ________

III-3. Coordinates in a Plane

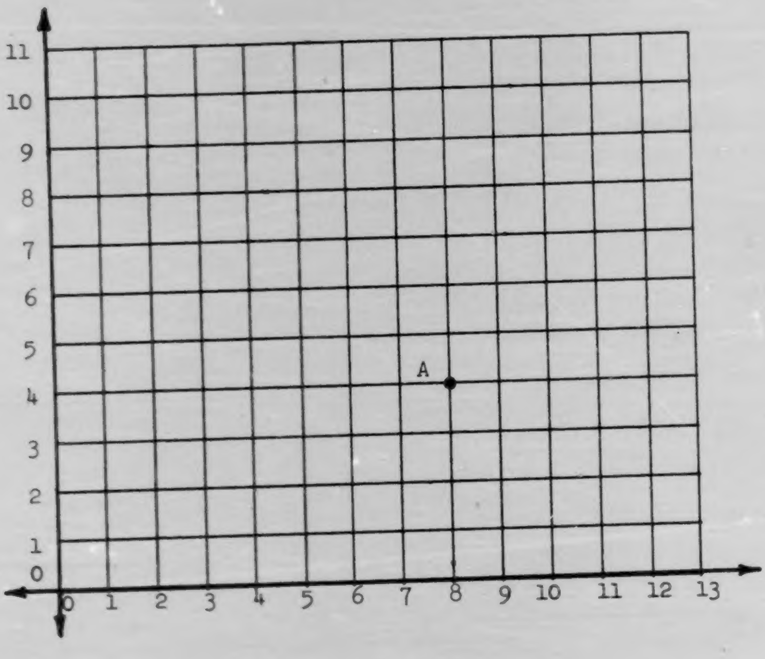
Objective: To use whole numbers to describe certain points in the plane.

Vocabulary: Coordinate axis, coordinate axes, origin.

Materials: Figure on chalkboard as indicated below.

Suggested Procedure:

Ask the children if they have ever seen a tile floor paved with square tiles, all the same size. (If the schoolroom is paved in this way it can be used as a specific illustration.) Have on the board a drawing like the one below (but without the numerals).



Ask the children if the figure shows how a floor looks when paved with congruent square tiles. (Note: There are other ways of paving with square tiles, but this is the pattern we support for use here.) Have the children look at the drawing and point out the way in which the square regions fit together, four at a point. Remind them that we know this is what happens with regions having right angles.

Suggest that we think of the lines at the bottom and at the left-hand side of the figure as the walls of a room and the square regions as tiles in the floor. (Note: If you are actually using the tiled classroom floor, the walls probably are not exactly on the lines formed by the edges of the tiles. In this case, use the lines formed by edges of uncut tiles which are closest to the walls.)

Recall with the class that for several days we have been using numbers to describe points on a line. Now we will think of points in a plane. Mark a point on the diagram, say the point A, and ask if anyone can think of a way of describing where that point is. A little discussion should elicit the idea that this point could be located by starting at the lower left corner and going 7 units to the right and 4 units up (or 4 units up and 7 over). That is, it takes two numbers to describe point A. Point A is described by the symbol (7, 4). The two numbers used to describe A are often called coordinates of A.

Notice that we agree that the first of the numbers will tell us how far to move to the right and the second how far up. Emphasize this by having some pupil locate the point (4, 7) and note that it is a different point.

Suggest that since to find the coordinates of a point (numbers describing a point) we have to count numbers of units to the right and up, it might be useful to mark number scales on the bottom line and the left-hand line. Have the numerals written on the figure as shown with the lower left-hand corner the zero point of both scales.

Point out that the two number scales are convenient as they save counting. If we follow down from A we find the first coordinate 7 and if we look to the left from A we find the second coordinate 4. (Note: The number line on which the first coordinate is found is called the first coordinate axis; that on which the second coordinate is found is called the second coordinate axis. Use of these two coordinate axes makes our present work easier. The point of intersection of the two coordinate axes is called the origin of coordinates or simply the origin.)

Mark other points and have children give the pair of coordinates. Then give pairs of coordinates and have children locate the points.

Pupil's book, pages

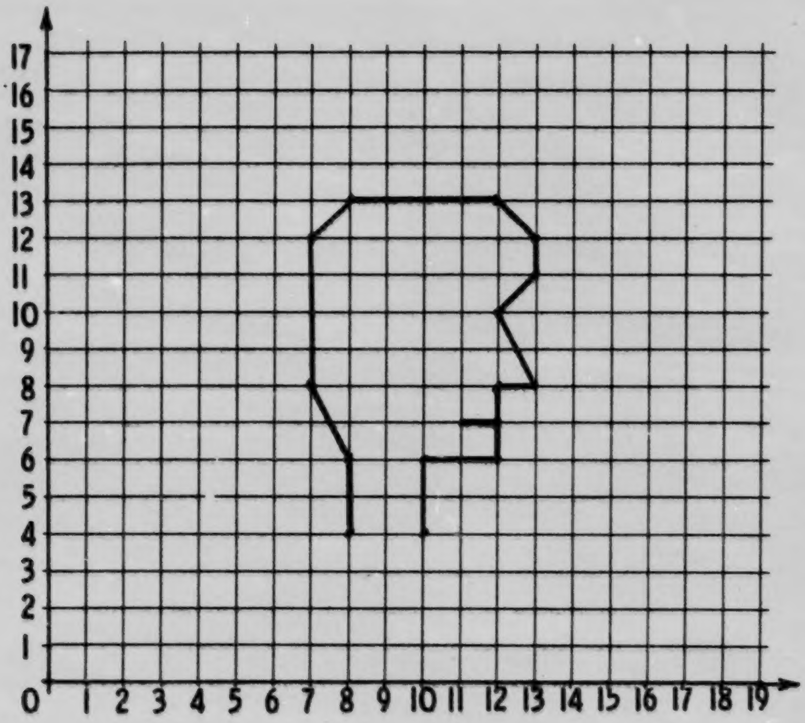
Children should now be ready for these pages. Show the children that, in problems 3 and 4, the problems are stated on one page and the drawings are on the facing page. Problem 4 is intended to review ideas of sets of points and may be better done working together. Problem 5 is intended to provide creative activity for pupils who are interested and able.

Pupils may again bring up the matter of rational and negative coordinates. If so, agree that this can be done and indicate that they will see this in the future. At the present we will use only whole numbers as coordinates. If some children wish to use these numbers (rational and negative) in independent activities, encourage them to do so.

Coordinates in a Plane

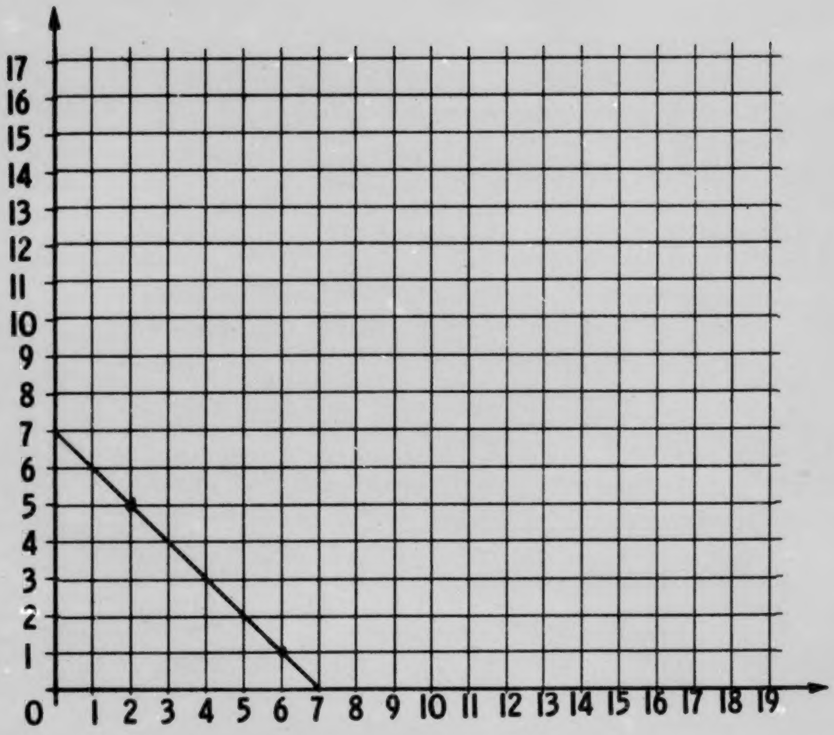
1. Draw segments joining the following points in order:

(10, 4) (10, 6) (12, 6) (12, 7) (11, 7) (12, 7) (12, 8)
(13, 8) (12, 10) (13, 11) (13, 12) (12, 13) (8, 13) (7, 12)
(7, 8) (8, 6) (8, 4).



What did you find? A face (or head).

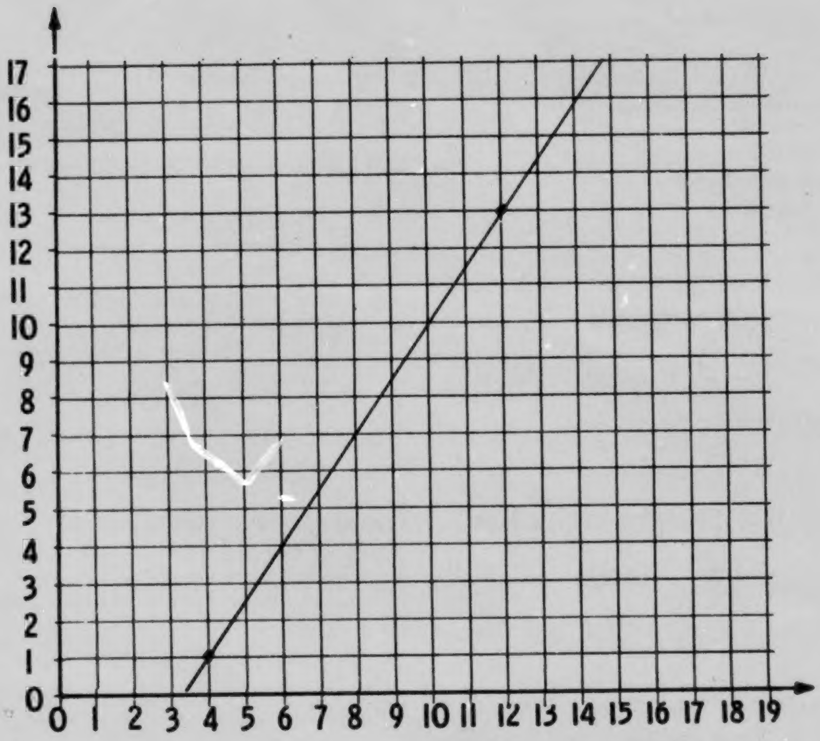
2.



Use your ruler to draw the line through the points (6, 1) and (2, 5).

Other points which seem to lie on this line are (0, 7), (1 , 6), (3 , 4), (4, 3), (5 , 2), and (7, 0).

3.



Use your ruler to draw the line through the points (4, 1) and (12, 13). Some other points which seem to lie on this line are (6,4) and (8,7), also (10,10) and (14,16).

4. Draw segments joining the following points in order :
(2, 13) (2, 9) (2, 11) (4, 11) (4, 9) (4, 13).

Draw segments joining the following points in order :
(7, 12) (5, 12) (5, 10) (6, 10) (5, 10) (5, 8) (7, 8).

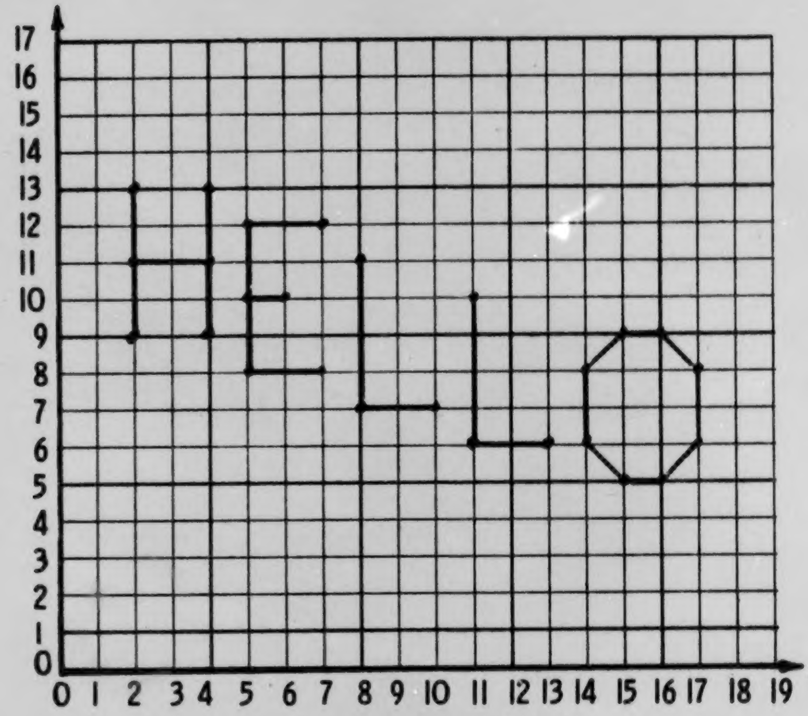
Draw segments joining the following points in order:
(8, 11) (8, 7) (10, 7).

Draw segments joining the following points in order:
(13, 6) (11, 6) (11, 10).

Draw segments joining the following points in order:
(15, 5) (16, 5) (17, 6) (17, 8) (16, 9) (15, 9)
(14, 8) (14, 6) (15, 5).

What did you find? A word — HELLO.

4.


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

5. Give numbers describing A, B, C, D.

A(0, 10), B(4, 12), C(8, 4), D(4, 2).

Draw $\overline{AC}$ and $\overline{BD}$. Call their point of intersection E.

Give numbers describing E. (4, 7)

The length of $\overline{BD}$ is 10 units.

Draw $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

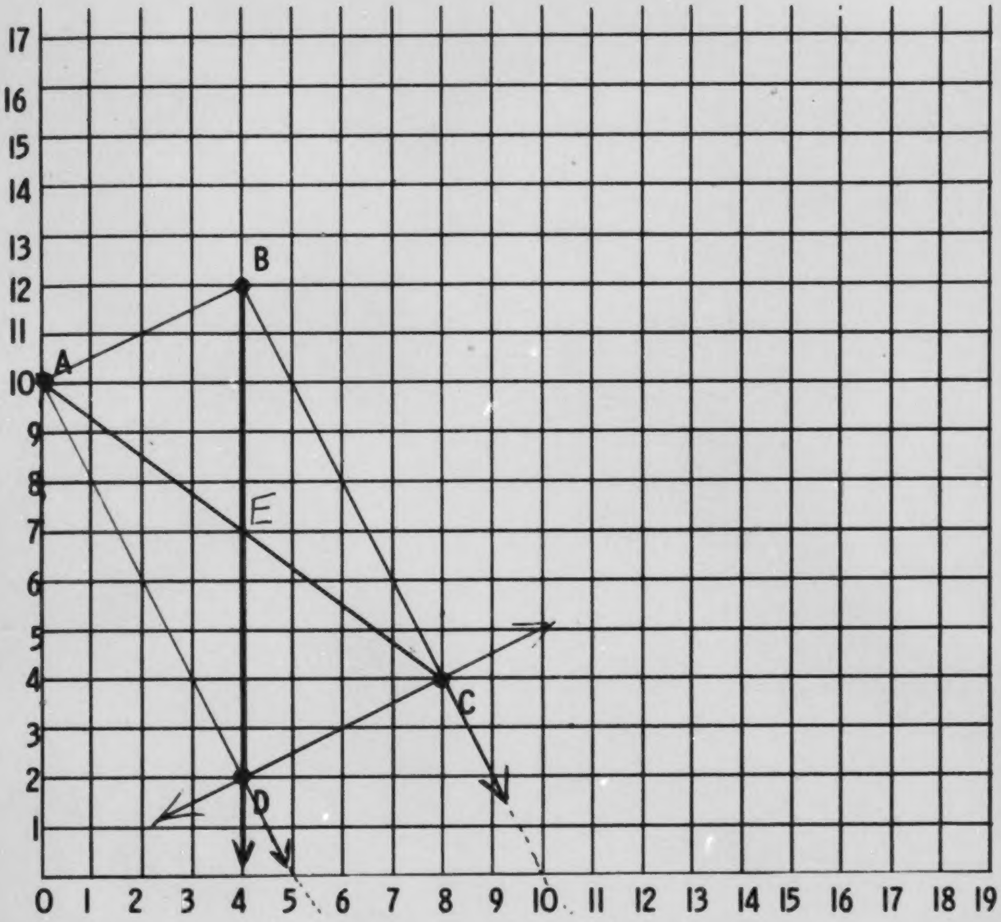
Give numbers describing the points where $\overrightarrow{AD}$ and $\overrightarrow{BC}$ meet the bottom line. (5, 0) (10, 0)

Draw $\overline{AB}$ and $\overline{CD}$. What kind of figure is ABCD? Rectangle

Draw $\overleftrightarrow{CD}$.

Give numbers describing the point where $\overleftrightarrow{CD}$ meets the bottom line. (0, 0).

5.


A
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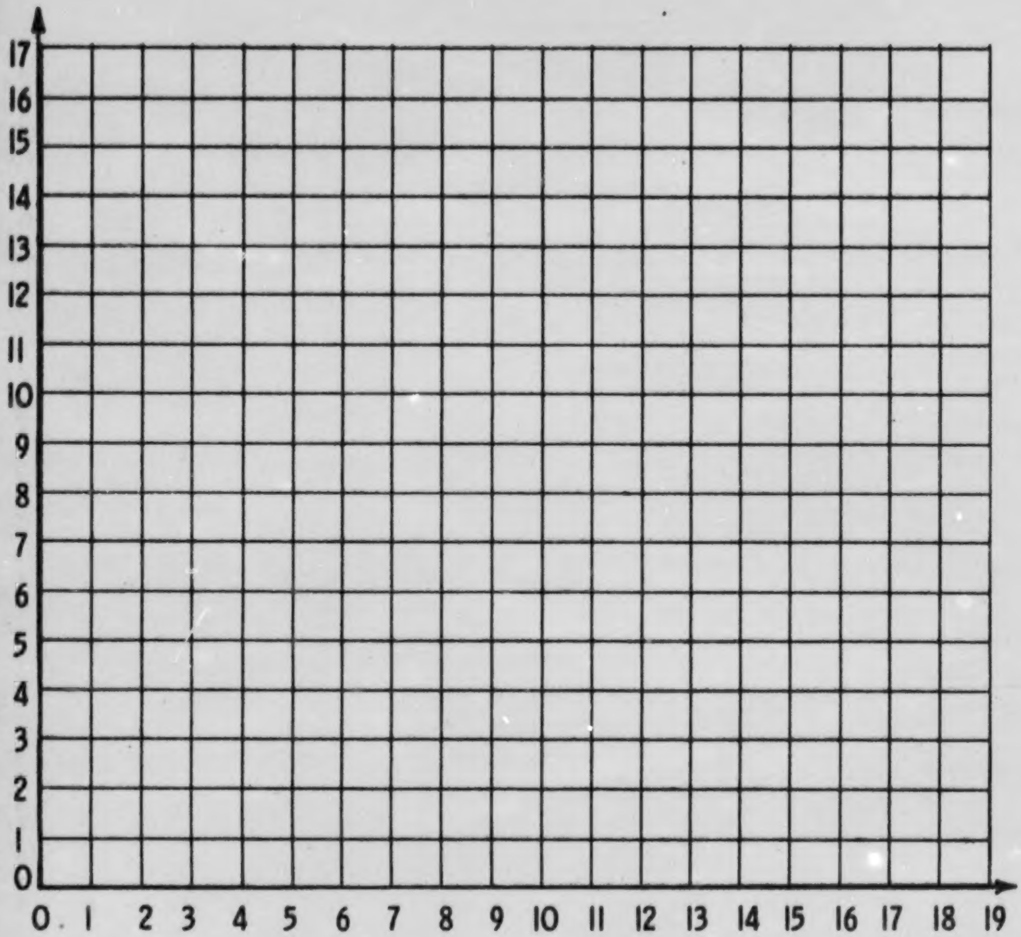
6. Make a figure on the facing page.

   Use only segments whose endpoints are described by whole numbers.

   Use the numbers to tell how to draw your figure.

   See if a classmate can follow your directions without seeing your figure.

6.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Answers will vary.

III-4. Plane Figures

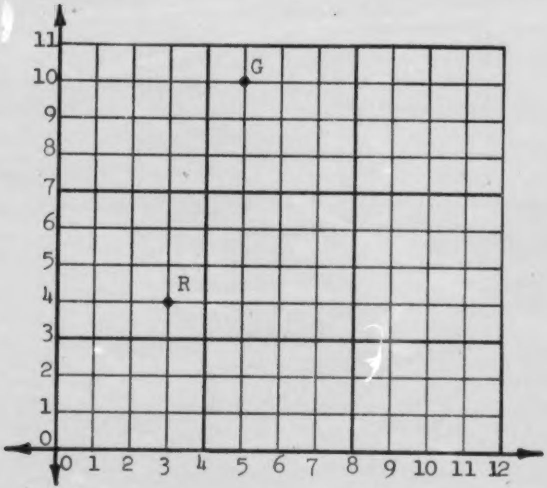
Objective: To observe the effect on plane figures when numbers are added to the coordinates of their points.

Vocabulary: (No new words.)

Materials: Coordinate system drawn on the board or on large sheet of paper (two such systems would help). Colored chalk, tracing paper, two small disks of different colored paper (e.g., red and green) with provision for attaching them at desired places of the coordinate system.

Suggested Procedure:

Tell the pupils we will begin this lesson with a game. Take the red and green disks and place them on the coordinate system at some points, perhaps those shown below where R indicates the red disk and G the green one. Explain that these disks may



be moved about the plane in certain ways. The red disk may be moved one unit horizontally (either right or left) and two units vertically (either up or down). The green disk on the other hand may be moved two units horizontally and one unit vertically. (Rules are displayed in the chart at the right.) We agree that neither is allowed to move off the part of the coordinate system we have drawn. The object of the game is to move the disks alternately in such a way that they land on the same point. We keep track of the play by recording the positions of the disks in the table below. The first line records the original position of the disks.

Rules:

	Red	Green
Right or Left	1 move	2 moves
Up or Down	2 moves	1 move

R	G
(3, 4)	(5, 10)

The red disk is moved first. There are four possible points where it can be moved. Let the class decide where to move and record the new position in the first column. Have the children notice that to get the new coordinates, one adds or subtracts 1 from the 3 and adds or subtracts 2 from 4. Then the green disk moves and its new position is recorded. Continue until the disks are on the same point. There is nothing unique about the play. For example, two successful ways are listed below. Notice that in the second case there were unnecessary moves since the position shown in the second and fourth rows are the same.

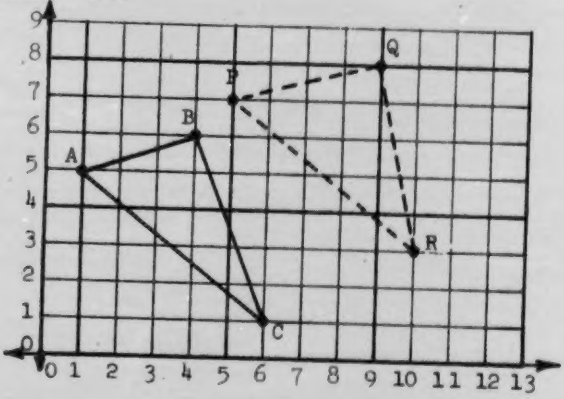
R	G
(3, 4)	(5, 10)
(4, 6)	(3, 9)
(5, 8)	(5, 8)

R	G
(3, 4)	(5, 10)
(2, 6)	(3, 9)
(3, 8)	(1, 8)
(2, 6)	(3, 9)
(1, 8)	(1, 8)

The game can be varied by changing the initial positions and changing the allowed moves. The pupils may be interested to discover that sometimes it is impossible to win. For example, suppose all the moves involve an even number of units both ways and suppose R starts on a point with both coordinates odd. Then all the points it can land on will also have both coordinates odd. If G starts with both coordinates even, then all the points it can land on will also have both coordinates even, so R and G can never get together.

Children may like to play this game by themselves later. Its purpose here is to develop the idea that a motion in a horizontal direction is associated with adding to or subtracting from the first coordinate, with a similar remark for the relation of vertical motion to the second coordinate. This is the reason for recording each move as noted above.

Now examine a coordinate system on the chalkboard on which you have drawn a triangle in colored chalk as shown below.



Notice the coordinates of the three points. In this case they are A (1, 5), B (4, 6), C (6, 1). From these three points A, B, C, find three other points P, Q, R by moving 4 units to the right and 2 units up. This is exactly the process the children have been doing in the game. Have them notice that it amounts to adding 4 to each first coordinate and 2 to each second coordinate. Have points P, Q, R marked on the coordinate system and have △ PQR drawn, preferably using a different color of chalk. This is shown above by the dotted line or segments.

Discuss with the class the relationship of the two figures. Specifically, consider whether the two triangles look to be congruent. Use a string or edge of a paper or some other means to check that $\overline{AB}$ does appear congruent to $\overline{PQ}$, similarly, $\overline{BC}$ appears to be congruent to $\overline{QR}$ and $\overline{AC}$ appears to be congruent to $\overline{PR}$.

Now distributie tracing paper and have the pupils turn to problem A on page 190 in the pupil's book. This problem is similar to what has just been done on the board, but tracing paper is used to verify the congruence. Remind the pupils again of the best way to make a tracing of the triangle. Have the children place the tracing over △ ABC. Try to have them see that if they slide the tracing 7 units to the right and 4 units up, the three dots on the tracing which were over A, B, C will then be over P, Q, R. This means that △ ABC is congruent to △ PQR.

On the chalkboard list the coordinates of the points like this:

A (2, 6)	B (6, 1)	C (3, 10)
P (9, 10)	Q (13, 5)	R (11, 14)

Then call attention to the fact that the first coordinates in the lower line are all seven greater than the corresponding first coordinates in the upper line.

Also, the second coordinates in the lower line are all 4 greater than the corresponding second coordinates in the upper line. Discuss with the class, with other examples if necessary, whether relations like this among coordinates will always indicate congruent figures. (The answer is "Yes".) An example is

A (2, 6)	B (6, 1)	C (3, 10)
L (5, 7)	M (9, 2)	N (6, 11)

Triangle LMN should be drawn now.

Pupil's book, pages 192-193 and 194-195:
This will help to verify the conjecture in case the two figures are not triangles.

Pictures in the Plane

A. Look at the figure on the next page.

The numbers describing A, B, C are

A(2 , 6), B(6 , 1), C(3 , 10).

Move 7 units to the right and 4 units up from each point.

Call these new points P, Q, R.

The numbers describing P, Q, R are

P(9 , 10) Q(13 , 5), R(10 , 14).

Mark P, Q, R.

Draw $\overline{PQ}$, $\overline{QR}$, and $\overline{RP}$.

Make a tracing of △ABC.

Does this tracing exactly fit on △PQR? Yes

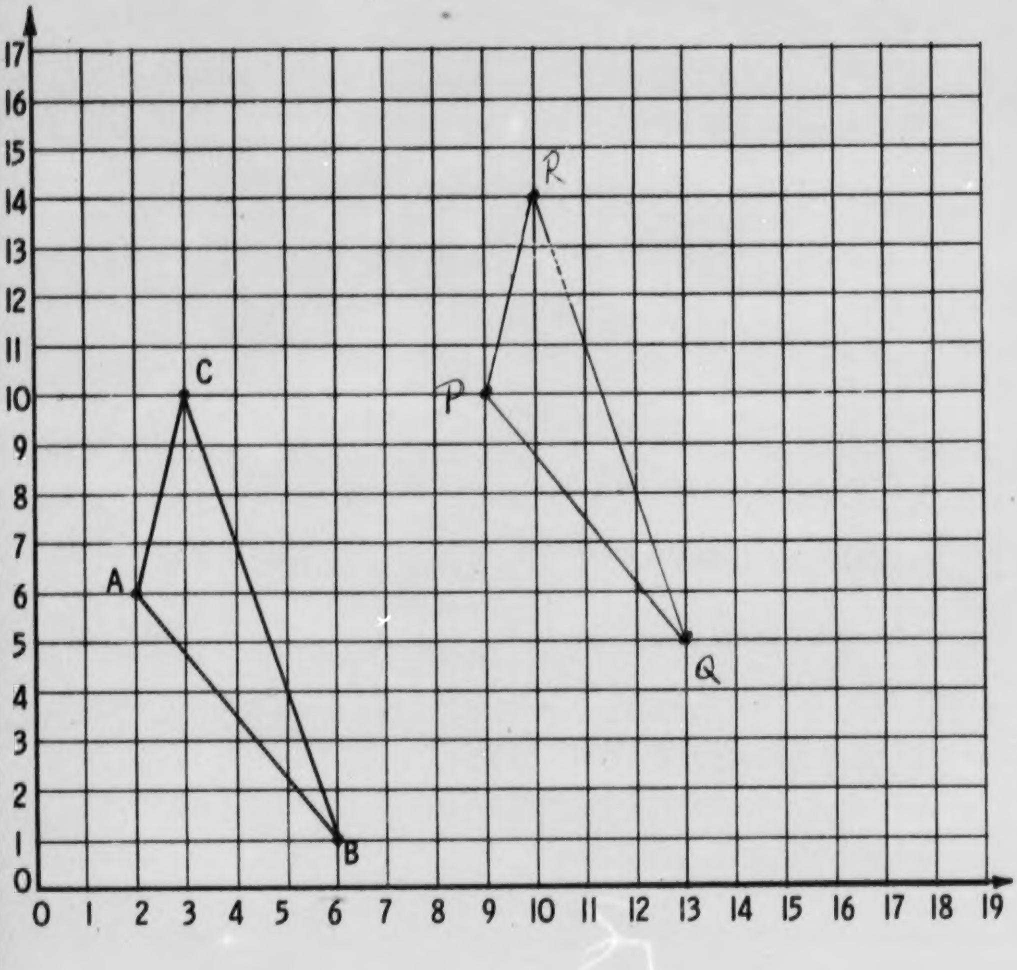
Do you find △ABC congruent to △PQR? Yes

Complete the table below to show congruent sides and angles.

$\overline{AB}$	$\overline{PQ}$
$\overline{BC}$	$\overline{QR}$
$\overline{AC}$	$\overline{PR}$
∠ABC	∠PQR
∠CAB	∠RPQ
∠BCA	∠QRP

Pictures in the Plane

A.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0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
A
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C
P
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R

Pictures in the Plane

1. The pairs of numbers describing A, B, C, D are
   A(2, 12) B(0, 7) C(7, 3) D(5, 9).

   Points P, Q, R, S are found by adding 5 to the first number in each pair.
   The second numbers are not changed.
   The pairs of numbers describing P, Q, R, S are
   P(7, 12), Q(5 , 7), R(12 , 3), S(10, 9).

   Mark A, B, C, D, P, Q, R, S on the opposite page.

   Draw quadrilateral ABCD.

   Draw quadrilateral PQRS.

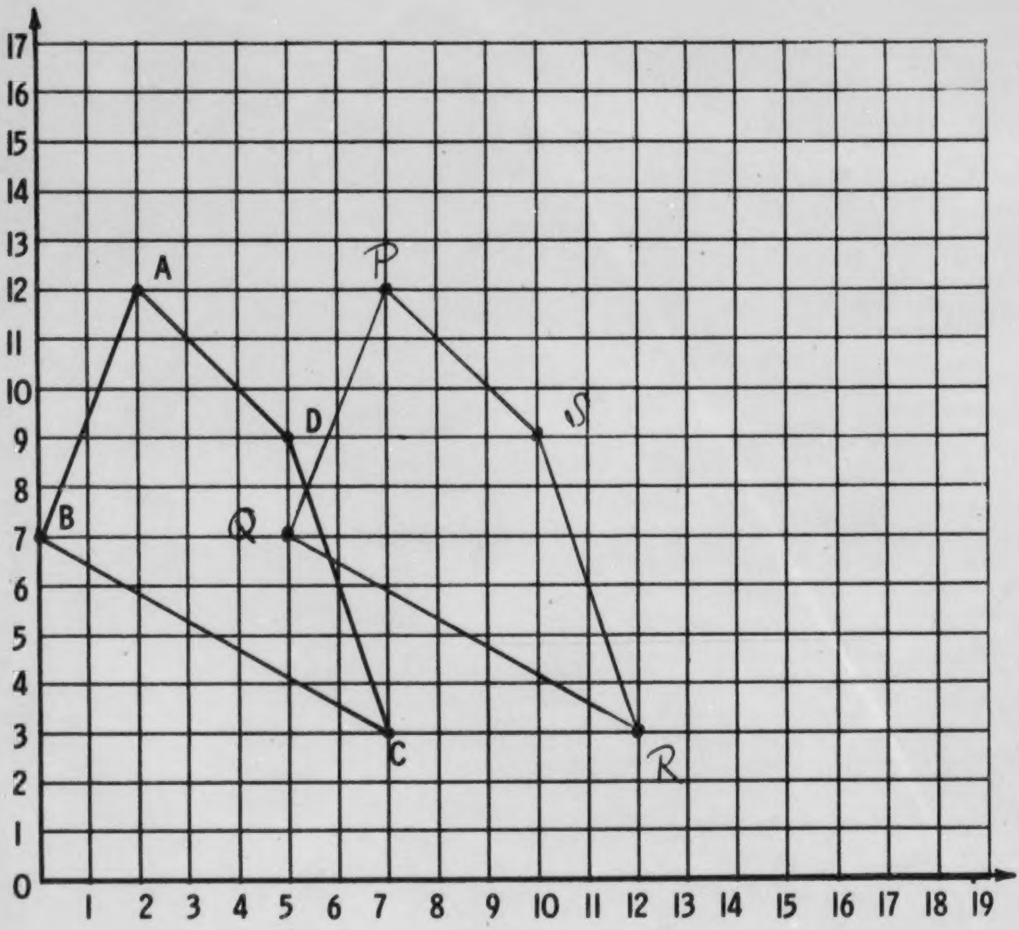
   Make a tracing of ABCD.

   Can you fit the tracing on PQRS? Yes

   Is ABCD congruent to PQRS? Yes

Pictures in the Plane

1.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2. The pairs of numbers describing A, B, C, D, E are

A(1, 9) B(5, 7) C(2, 2) D(11, 1) E(6, 13).

Points P, Q, R, S, T are found by adding 6 to the first number in each pair and 2 to the second number.

The pairs of numbers describing P, Q, R, S, T are

P(7, 11), Q(11, 9), R(8, 4),

S(17, 3), T(12, 15).

Mark all these points on the opposite page.

Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EA}$.

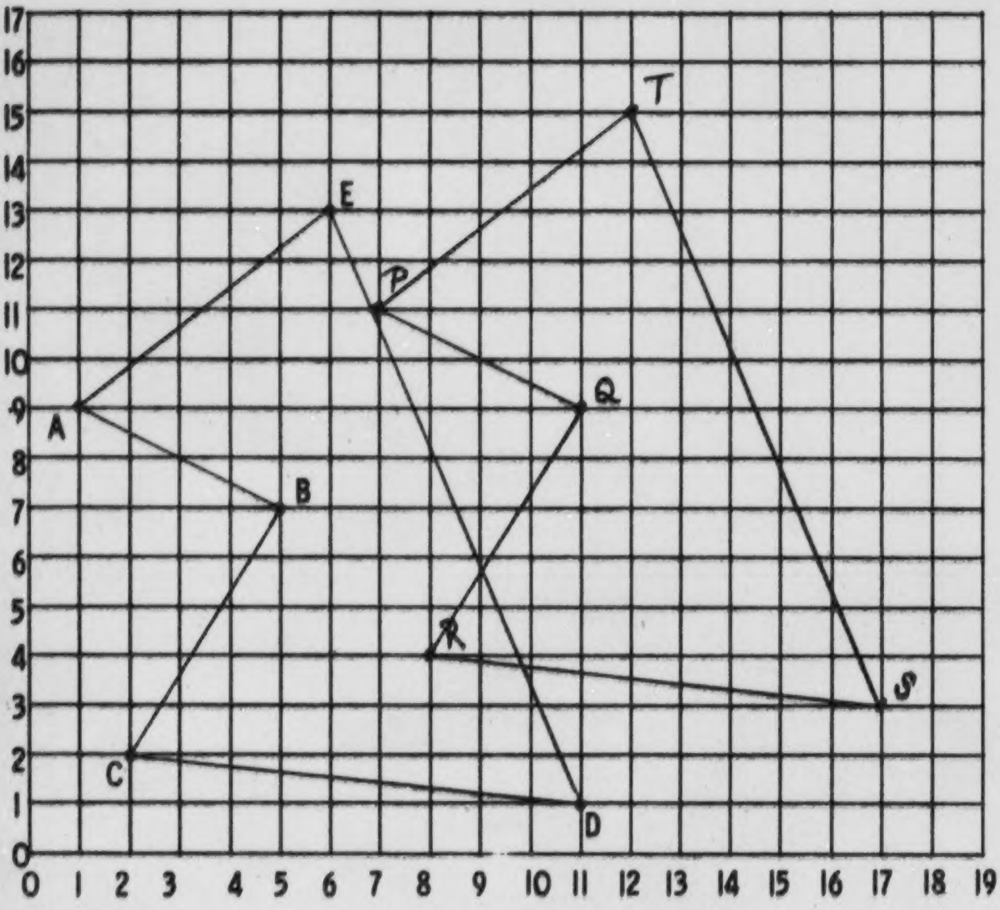
Draw $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, $\overline{ST}$, $\overline{TP}$.

Make a tracing of ABCDE.

Can you fit the tracing on PQRST? Yes

Is ABCDE congruent to PQRST? Yes

2.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III-5. Stretching Pictures of Segments on a Line

Objective: To observe the effect when the coordinate of each end of a segment on a line is doubled or tripled.

Vocabulary: (No new words.)

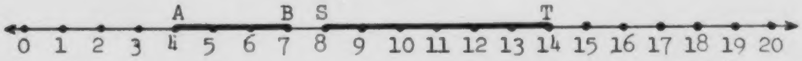
Materials: Colored chalk.

Suggested Procedure:

You may wish to postpone these sections!

(Note: The last three sections of this chapter will make use of the idea of doubling or tripling the coordinates of points, that is, multiplying them by 2 or 3. These sections should not be used until children have understanding and some skill in multiplication. You may wish to use these sections after children have had more work in multiplication.

It will be well to have three or four number lines already drawn on the board. On one of them mark a segment, say $\overline{AB}$ as shown. Indicate it with colored chalk.



Have the coordinates of A and B noted as 4 and 7. Recall with the class that in section 2 we noticed what happened if the same number was added to the coordinates of the endpoints of a segment.

What was the result? (The new segment was congruent to the original.)

This time we shall try to see what happens if each of the coordinates is multiplied by the same number. Specifically, let us double each coordinate, that is, let us multiply each coordinate by 2. If the coordinate of A is doubled, we get 8. Let S be the point with coordinate 8. Similarly, let T be the point whose coordinate is 14 which is 2 times the coordinate of B. Let these points be marked on the line by some of the children. Then have $\overline{ST}$ colored, using some other color of chalk.

Now examine these segments with the class. Recall that in Section 2 the new segment was always congruent to the old one. Ask whether this is true here, i.e., is $\overline{ST}$ congruent to $\overline{AB}$? (No.)

Ask children to suggest how they think $\overline{ST}$ and $\overline{AB}$ are related. It is certainly easy to observe that in this case the length of $\overline{ST}$ (14 - 8 = 6.) is just twice that of AB (7 - 4 = 3). Whatever conjectures are offered, have them tried out using different segments on the other number lines on the board. This should lead to confirming the idea that when the coordinate of each end of a segment on a line is doubled, the length is doubled. Be sure to take some cases where the two intervals overlap, as in the case of points with coordinates 2 and 6 where the doubled coordinates are 4 and 12. It will be desirable to discuss with the class the reasons why they think it reasonable to guess that the new segment is always twice as long as the original one. For instance, some children may make the observation that since each point is matched up with one twice as far from 0, it looks quite likely that the distances are multiplied by 2. (Actually this remark is the basis of a proof. For example, in the figure above the length of $\overline{AB}$ is (7 - 4) units. The length of $\overline{ST}$ is similarly $(2 \times 7) - (2 \times 4)$ units. But $(2 \times 7) - (2 \times 4) = 2 \times (7 - 4)$ so $\overline{ST}$ has twice the length of $\overline{AB}$. This, however, used the distributive property of multiplication with respect to subtraction, a fact not formally discussed with the pupils. This discussion provides you with additional background.

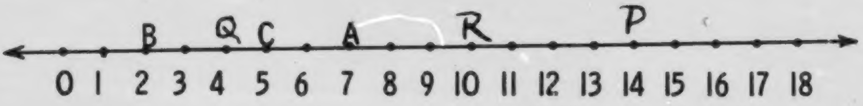
Do not use it with children. Take examples where the coordinates of the endpoints of a segment are multiplied by some other number, say 3. Help children to discover that in these instances the one segment will be 3 times as long as the other segment. Then verify that it works in one or more cases. For example, take the coordinates of C and D to be 2 and 5 respectively. Multiply each by 3 to get the coordinates of U and V to be 6 and 15 respectively. The length of $\overline{CD}$ is (5 - 2) or 3 units, while the length of $\overline{UV}$ is seen to be (15 - 6) or 9 units. We then see that the measure of the length of $\overline{UV}$ (9) is three times the measure of the length of $\overline{CD}$ (3). Hence, if we multiply the coordinate of each endpoint of $\overline{CD}$ by 3 we produce the coordinates of the end points of a new segment $\overline{UV}$. To find the number of units of length for $\overline{UV}$, we multiply the number of units of length for $\overline{CD}$ by the same number, by the same number, 3.

By this time you may find the class ready to guess that if the coordinates of the endpoints of a segment are each multiplied by some number k, then the length of the new segment will be k times that of the original. Don't push for a specific verbalization, but try to make the idea as clear as possible. You can tell children that this guess is actually a correct one, though of course we have not really proved it.

Pupil's book, pages 197 and 198 : These pages illustrate the conclusions indicated here.

Enlarging Segments on the Number Line.

1. Points A, B, C are shown on the number line.



The number describing point P is two times the number for A.
The number describing point Q is two times the number for B.
The number describing point R is two times the number for C.

Mark points P, Q, R on the line.

Show below the number describing each point.

A	B	C	P	Q	R
7	2	5	14	4	10

Show below the number of units in each segment.

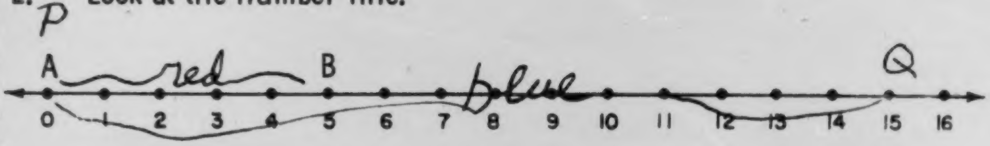
$\overline{AB}$	$\overline{BC}$	$\overline{AC}$	$\overline{PQ}$	$\overline{QR}$	$\overline{PR}$
5	3	2	10	6	4

Is $\overline{PQ}$ twice as long as $\overline{AB}$? Yes

Is $\overline{QR}$ twice as long as $\overline{BC}$? Yes

Is $\overline{PR}$ twice as long as $\overline{AC}$? Yes

2. Look at the number line.


P
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Color $\overline{AB}$ with a red crayon.

Multiply the numbers describing A and B by 3.

These new numbers are 0, 15.

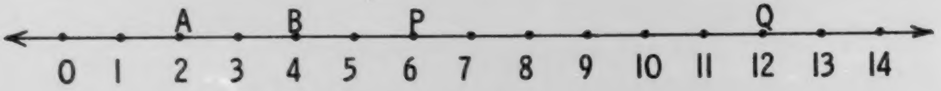
Call the new points P and Q.

Mark P and Q on the line.

Color $\overline{PQ}$ with a blue crayon.

The length of $\overline{PQ}$ is 3 times the length of $\overline{AB}$.

3. Look at the number line.


A
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The number describing P is 3 times the number describing A.

The number describing Q is 3 times the number describing B.

The length of $\overline{PQ}$ is 3 times the length of $\overline{AB}$.

Are the three numbers you wrote in the blanks the same? Yes

III-6. Enlarging Pictures

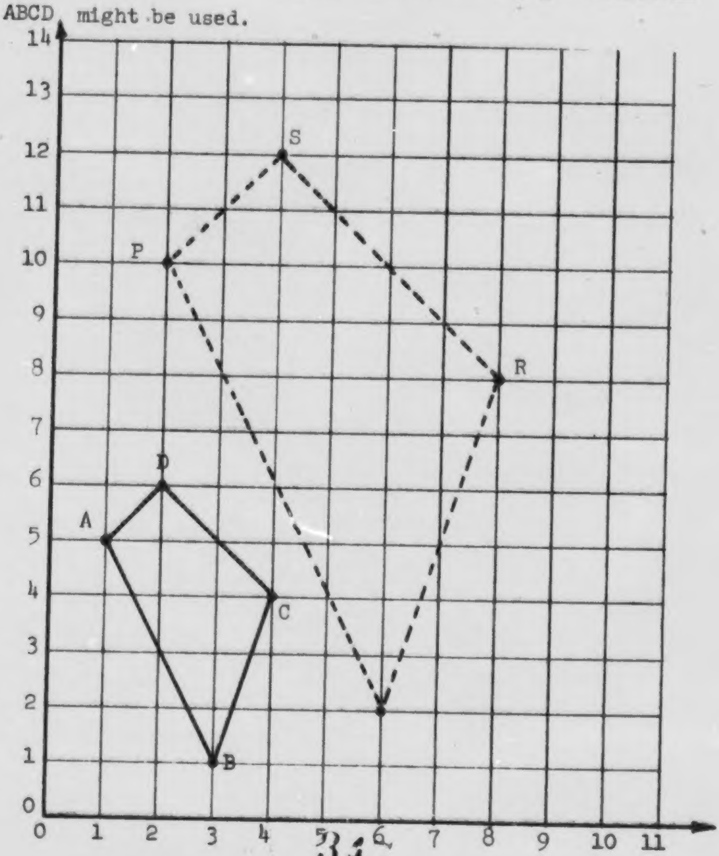
Objective: To see the effect on plane figures if all coordinates are multiplied by the same number.

Vocabulary: Similar.

Materials: A coordinate system drawn on the chalkboard (or on a large paper); colored chalk, tracing paper.

Suggested Procedure:

Have a figure drawn in colored chalk on the coordinate system. For example, the following quadrilateral ABCD might be used.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Have the pupils identify the coordinates as A (1, 5), B (3, 1), C (4, 4), D (2, 6). Suggest that they find four new points P, Q, R, S by multiplying all the coordinates by 2. Have the coordinates of P, Q, R, S found and have pupils plot them on the coordinate system. The quadrilateral PQRS can then be drawn in some contrasting color of chalk. (The result is shown above in dotted lines.) Be sure that the pupils see clearly that point P comes from A, Q from B, R from C, and S from D. This is important so that children will see clearly which are the corresponding parts of the two figures.

Have the class discuss the relationship between ABCD and PQRS. If a pupil suggests they "look alike" try to get him to be more precise as to what he means. Ask whether the figures appear to be congruent. (Clearly the answer is No.) On the basis of the work in Section 5 someone may conjecture that the sides of PQRS are twice as long as the corresponding sides of ABCD. Suggest this idea if it does not occur naturally. Have this conjecture tried out by some pupil. That is, mark off $\overline{AB}$ on the edge of a sheet of paper and see if this can be fitted just twice onto $\overline{PQ}$. Do the same for the other pairs of matching sides. It might even be interesting to have $\overline{AC}$ and $\overline{PR}$ drawn and check that $\overline{PR}$ is twice as long as $\overline{AC}$. (Note that these comparisons are most easily (and convincingly) made directly as indicated rather than by measuring the two segments and comparing results.

If, in the discussion of the figures, some pupil suggests that it looks as if the angles of the two figures are congruent, indicate that this is an interesting possibility that we will consider in our work in the pupil books. It is barely possible that some pupil who has run into the idea before may suggest that corresponding sides of the two figures are parallel. (We have not formally introduced the word, but two lines are *parallel* if they lie in a plane but do not have a point in common. Two *segments* are parallel if they lie on parallel lines.) This is a suggestion and is actually correct. You will not want to take a lot of time on it, but you could try out extending the segments as far as the chalkboard or paper will permit to see if it looks plausible that these lines have no point in common. One more suggestion that a bright pupil might make is that the lines joining any two corresponding points, i.e., AP, BQ, CR, DS all pass through the point (0, 0). This can be checked by laying a yardstick (or any straight stick) on the points to see. Even a piece of string would serve the purpose.

Pupil's book, pages 200-205: After discussing the figures on the board, turn with the pupils to pages . in the pupil's book. Distribute tracing paper. Because of the extended series of instructions this probably will need to be done together. The next problem on pages 202 and , 203 in the pupil's book is of the same nature but with a different figure.

From this experience it is hoped the pupil can be led to conjecture that whenever one figure is obtained from another by multiplying the coordinates by some number k, each segment of the new figure will be k times as long as the corresponding segment of the first figure, and that the angles of the two figures are congruent. This conjecture is correct.

Two figures which have these properties are called *similar*. The pupil has in this lesson seen one example of *similar triangles* and two examples of *similar quadrilaterials*.

Pupil's book, pages 204-205:

1. Have the pupils enlarge the small figure on page 204 of the pupil's book by multiplying all coordinates by 2.

2. Some pupils may enjoy drawing figures of their own on coordinate systems and enlarging them using this method. Pages 196-199 in the pupil's book may be used for this kind of activity.

3. Since in Section 4 we added different numbers to the first and second coordinates of the points of a figure, some pupils may be interested to see what happens if we multiply the first and second coordinates by different numbers. For example, in the drawing on page 201 of the pupil's book let the pupil multiply all first coordinates by 2 and all second coordinates by 1. Or again he might multiply all first coordinates by 1 and all second coordinates by 2. Pages 206-207 in the pupil's book may be used for such experiments. The pupil will discover that this time the new figure is not similar to the old one, but is distorted by a stretching in only one direction.

4. If a line is drawn from B (3, 1) upward, it will intersect the segment $\overline{DC}$ at the point, (3, 5). The distance from B to this intersection is 4 units. If a line is drawn from Q (6, 2) upward, it will intersect the segment $\overline{SR}$ at the point (6, 10). The distance from Q to this point is 8 units. Similarly, the measure of the length of the segment from D downward to $\overline{AB}$ is twice the measure of length of the segment from S downward to $\overline{PQ}$.

5. What generalization can be made from 4 above? (All corresponding distances are doubled.)

Enlarging Pictures

A. Look at the figure on page 201 .

The coordinates of A, B, C are

A(1 , 6), B(8, 2), C(4, 8).

Multiply all the numbers by 2.

Call the new points S, T, W.

The coordinates of S, T, W are

S(2, 12), T(16, 4), W(8 , 16).

Mark the points S, T, W.

Draw $\triangle$STW.

Draw a ring around each correct answer below.

Is $\overline{ST}$ twice as long as $\overline{AB}$? [✓ Yes] No

Tell how you found out.

Is $\overline{SW}$ twice as long as $\overline{AC}$? [✓ Yes] No

Is $\overline{WT}$ twice as long as $\overline{CB}$? [✓ Yes] No

Make a tracing of $\triangle$ABC.

Is $\triangle$ABC congruent to $\triangle$STW? Yes [✓ No]

Is $\angle$TSW congruent to $\angle$BAC? [✓ Yes] No

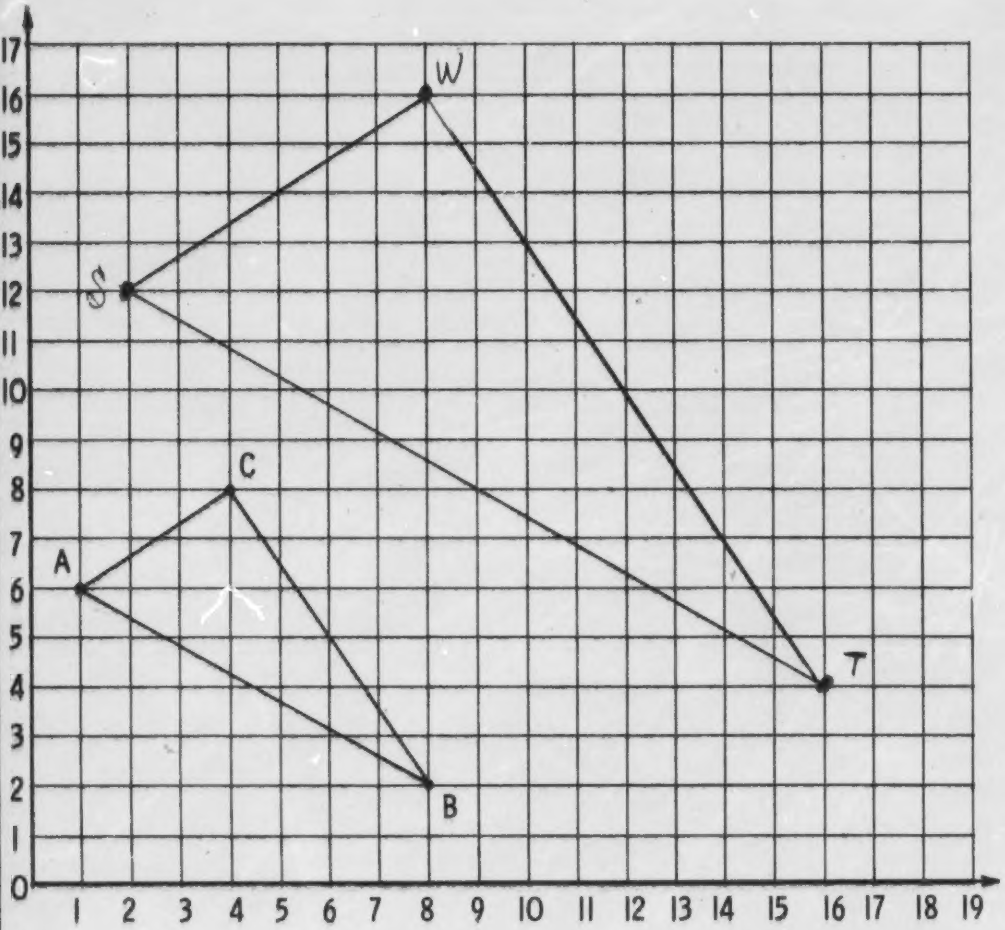
Use the tracing to check.

Name the angle congruent to $\angle$ABC. $\angle$ STW

Name the angle congruent to $\angle$TWS. $\angle$ BCA

Enlarging Pictures

A.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B. Look at quadrilateral ABCD.

Multiply all coordinates of these points by 3.

Call the new points P, Q, R, S.

The coordinates of P, Q, R, S are

P(0, 3), Q(9, 15), R(9, 6), S(18, 0).

Locate points P, Q, R, S.

Draw quadrilateral PQRS.

Is $\overline{PQ}$ three times as long as $\overline{AB}$? [✓ Yes] No

Is $\overline{QR}$ three times as long as $\overline{BC}$? [✓ Yes] No

Is $\overline{RS}$ three times as long as $\overline{CD}$? [✓ Yes] No

Is $\overline{PS}$ three times as long as $\overline{AD}$? [✓ Yes] No

Make a tracing of ABCD.

Is the angle at A congruent to the angle at P? [✓ Yes] No

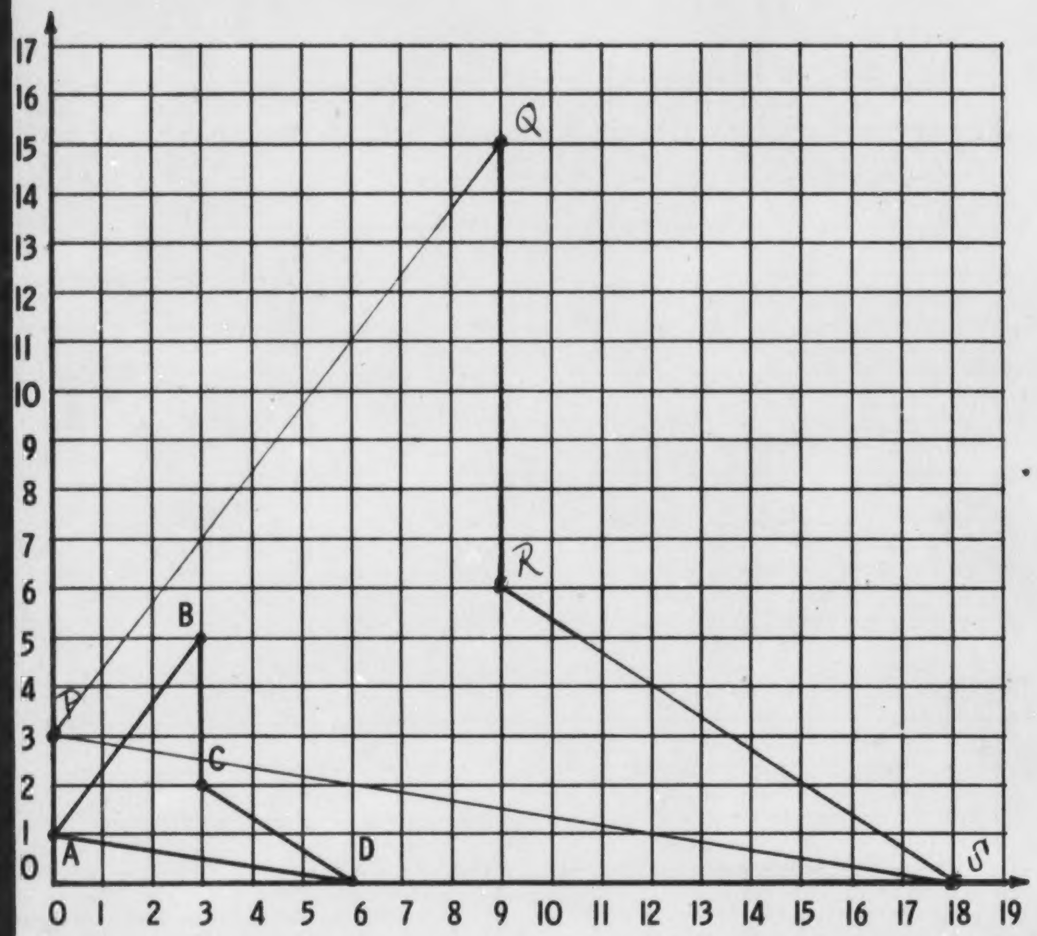
Use the tracing to find out.

The angle at B is congruent to the angle at Q.

The angle at S is congruent to the angle at D.

The angle at R is congruent to the angle at C.

B.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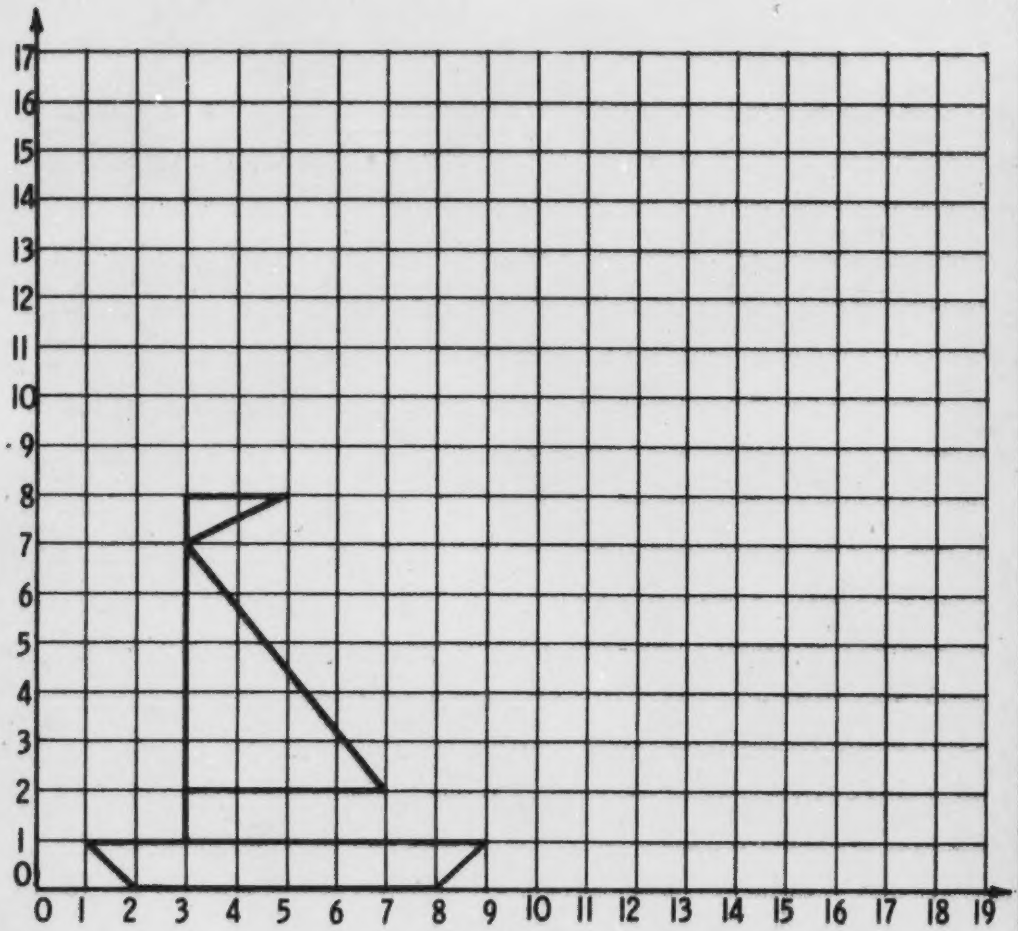
C.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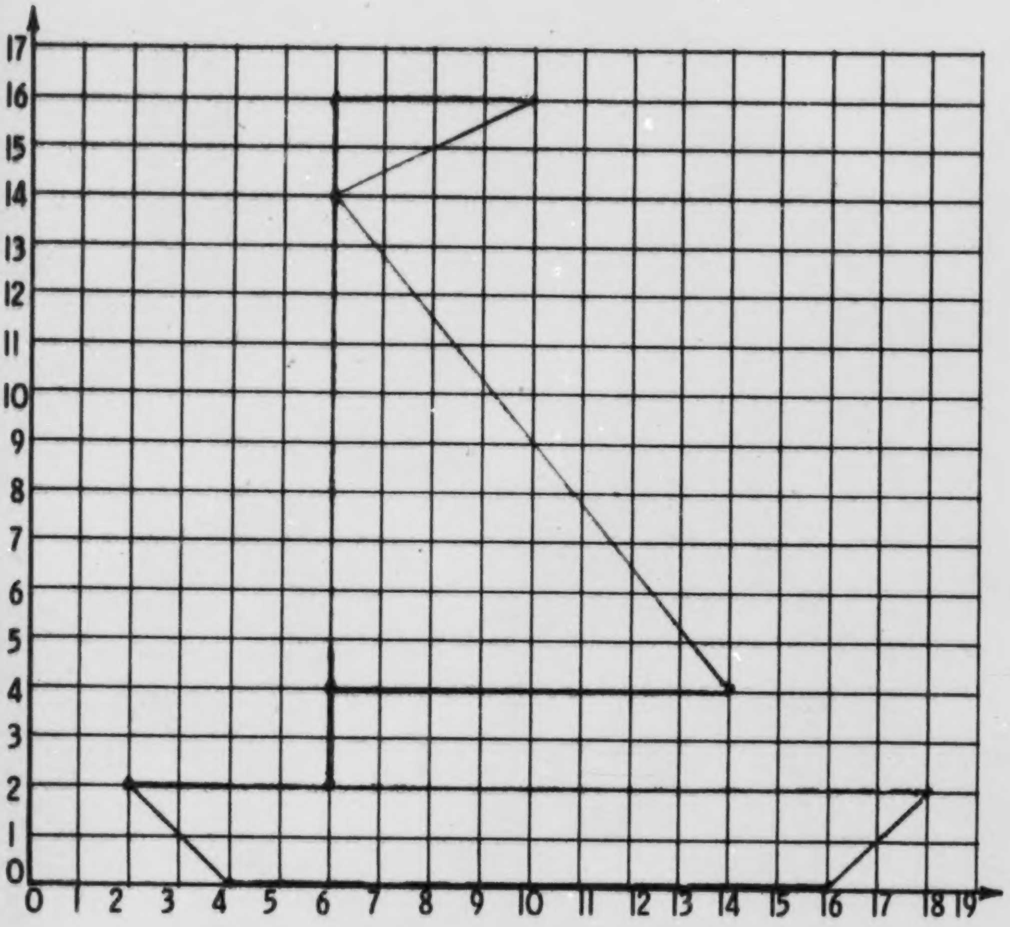
Make a larger picture of the boat on the facing page.

Multiply all coordinates by 2.

C.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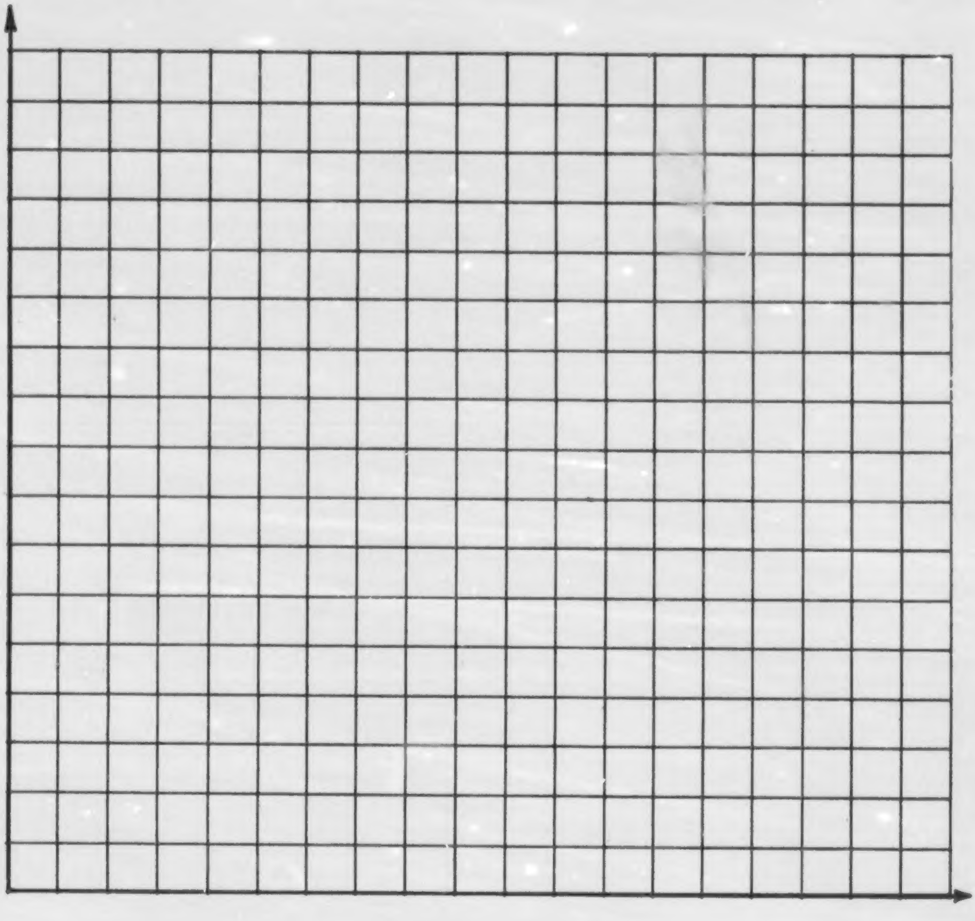
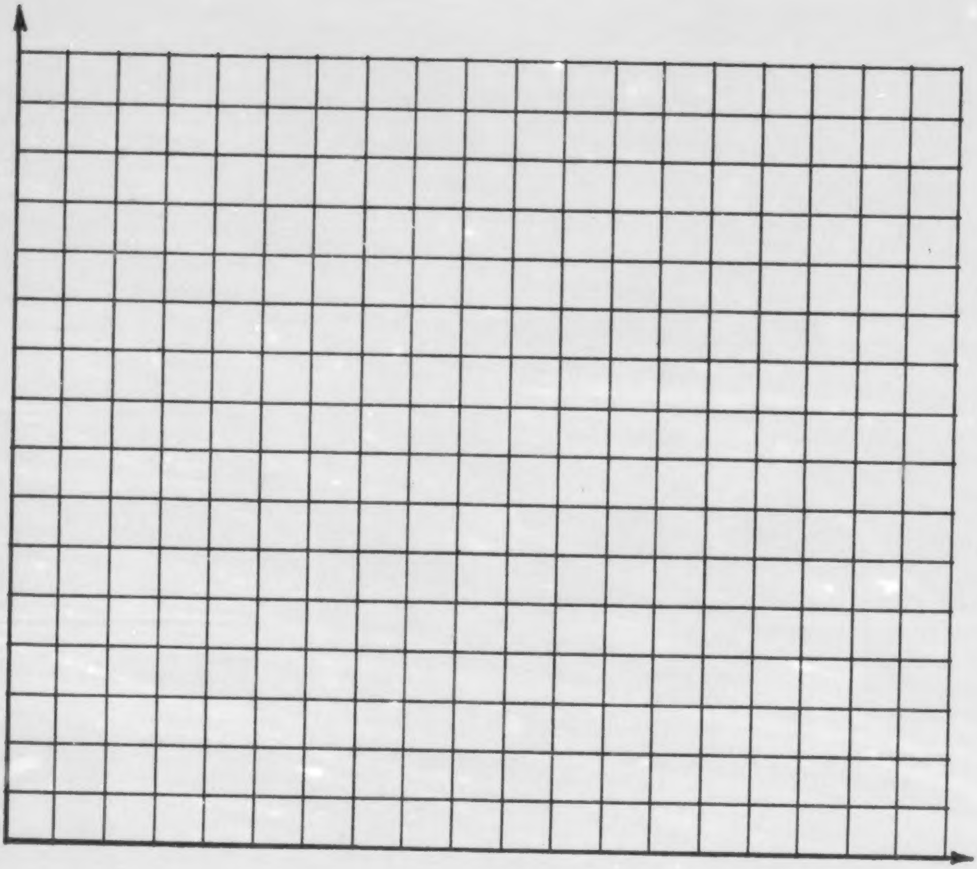




III-7. Scale Drawing

Objective: To introduce scale drawings.

Vocabulary: Scale drawing.

Materials: Yardstick, string, paper.

Suggested Procedure:

Remind the pupils that in the last lesson they saw examples of figures where any segment in one figure was matched with a segment twice as long in the other. Thus a segment one inch long of one figure corresponded to a segment two inches long in the other.

Tell the children that today we are going to make a drawing of the schoolroom floor so that a length of one inch on the drawing will correspond to a length of one foot on the floor. Have the children measure two sides of the room (to the nearest foot). Suppose that your floor turns out to be 20 feet by 30 feet. With the children's help, have a segment 30 inches long drawn on the chalkboard and explain that this is a drawing of one edge of the floor. Then make the drawing for the second edge 20 inches long. Be careful to make the angle a right angle. Then complete the rectangle. Label vertices A, B, C, D for convenience. Mark the segments as shown.

Bring out by questioning that the segments marked are not themselves 30 feet and 20 feet long.

> What do these markings mean? (That they are drawings of segments that have these lengths.) How long actually are the segments? (30 inches and 20 inches.)

Observe that the remaining segments $\overline{BC}$ and $\overline{CD}$ may also be marked 20 ft. and 30 ft. respectively.

Why? (Because the opposite sides of a rectangle are congruent.) Identify with the pupils which side of the rectangle is a drawing of which side of the floor. For example, $\overline{CD}$ may be identified as a drawing of the side of the classroom where the door is. You may even mark the location of the door if you wish. There is, then, a definite corner of the floor corresponding to each of the points A, B, C, D. Be sure this pairing is understood.

Now discuss with the class the question of how far it is diagonally across the classroom floor, say from the corner corresponding to C to the corner corresponding to A. Ask if they can think of any way to use the drawing to answer this question. Try to lead them to the idea of drawing $\overline{AC}$ and measuring this. If this does not come spontaneously, such questions as the following might help.

> We want the length of what segment? What represents this segment in the drawing? ($\overline{AC}$)
>
> Could you find the length of $\overline{AC}$? (Yes, by measuring.)
>
> If you know the length of $\overline{AC}$, can you tell the length of the segment on the floor? (Yes. One foot on the floor for each inch on the drawing.)

In the illustration given here $\overline{AC}$ is approximately 36 inches long so the diagonal distance across the floor is approximately 36 feet.

Having reached this conclusion, test it by actually measuring the distance on the floor. It is suggested that the best way to do this is to stretch a string across the floor from corner to corner and then measure the string.

Tell the children that a drawing such as we have made here in which a given length on the drawing always corresponds to a different given length in the original figure is called a _scale drawing_. Thus, on the board we have a scale drawing of our classroom floor.

(Note: The actual making of a scale drawing, if it is to be done with sufficient accuracy to be usable, would be quite demanding and tedious for most children at this grade level. Hence, the work which follows is based on a scale drawing which is already given.)

Pupil's book, pages   and
Follow the directions. This will probably need to be done together. Notice that this process is really just a case of measurement in a non-standard unit. For example, we find it takes 16 congruent copies of a given unit to cover $\overline{AB}$. This is precisely what it means to measure $\overline{AB}$ in this unit. Then we conclude that the segment in the room which corresponds to $\overline{AB}$ has a length of 16 feet.

Now observe that one of the common kinds of a scale drawing is a map. The problem on pages
in the pupil's book is an example of the use of a scale of miles on a map.

Further Activities:

1. Get a highway map of your state or section. Such a map will have a scale of miles similar to those on page   in the pupil's book.

This can be used in a similar way, but this time it can be used on cities which the children know, at least by name. Many roadmaps also have a chart showing distances between major cities. This will provide an interesting check for some of the distances the children obtain by measurement. Note that there may not always be agreement since the distances recorded on the chart will be road distances while those obtained by direct measurement are airline distances. Also, for long distances the curvature of the earth and the fact that we are working with flat maps will contribute to inaccuracies.

Reading Scale Drawings

1. Look at the figure on the facing page.

   ABCD is a scale drawing of the floor of a room.

   PQRS shows a table in this room.

   See the scale below the picture.

   Each small segment of this scale shows a one-foot segment in the room.

   Lay off this scale on the edge of a piece of paper.

   Lay it off several times to make a scale at least 20 units long.

2. Use the scale to find the following distances in the room (to the nearest foot).

   Length of longer side 16 ft.

   Length of shorter side 10 ft.

   Longer side of table 4 ft.

   Shorter side of table 2 ft.

   Distance matching $\overline{DB}$ 19 ft.

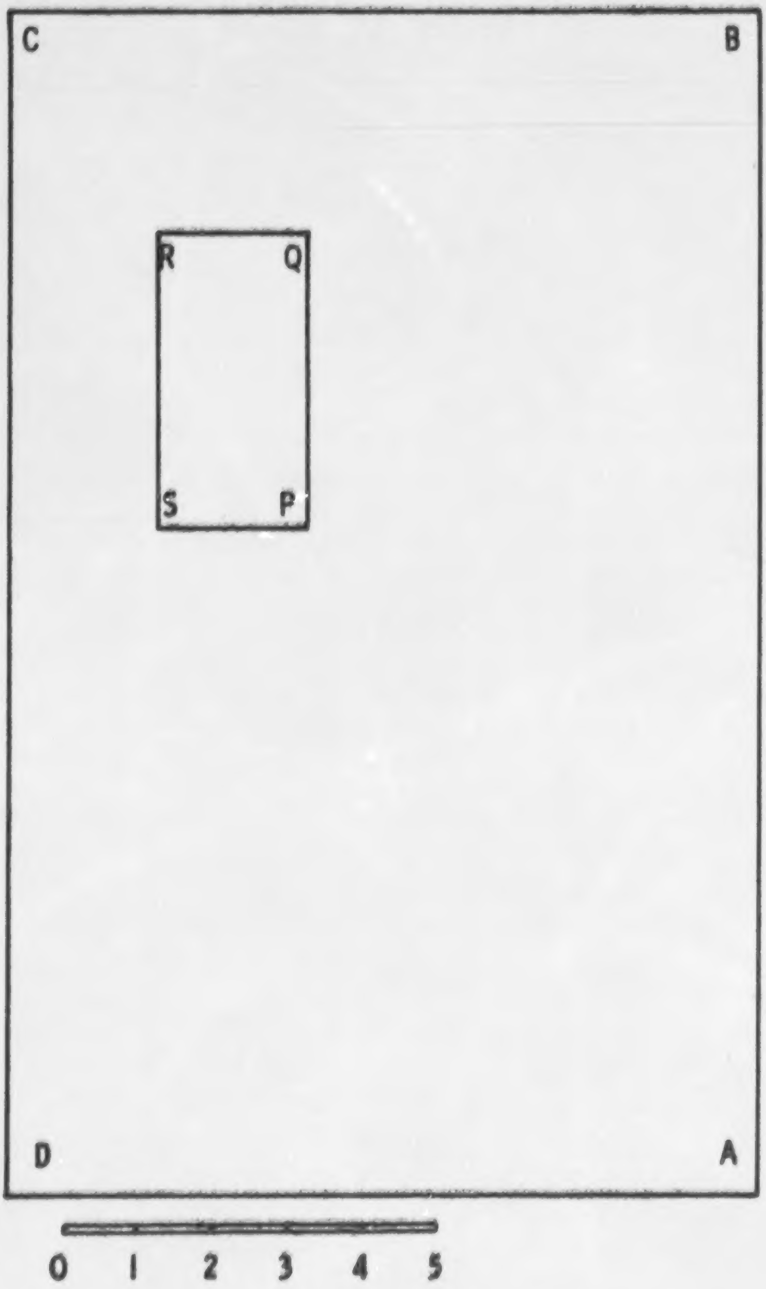
   Distance from the point matching C to nearest corner of the table

   4 ft.

   Distance from the point matching C to farthest corner of the table

   8 ft.

Reading Scale Drawings


C
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Look at the figure on the facing page.

It is part of a map.
See the scale below the map.
Each little segment on this scale stands for one mile.

Find the following distances:

Shortest distance from Madison to Conway is 7 miles.

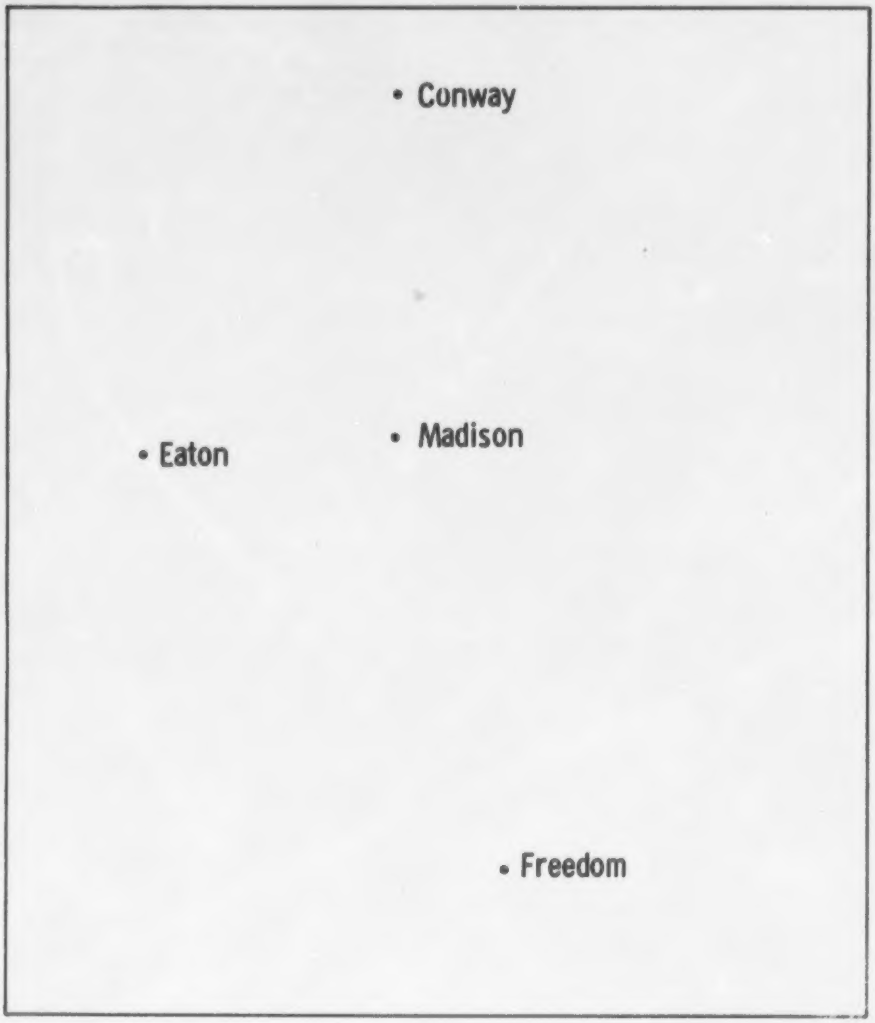
Shortest distance from Madison to Eaton is 5 miles.

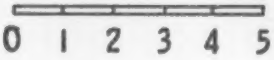
Shortest distance from Madison to Freedom is 9 miles.

Shortest distance between Freedom and Eaton is 11 miles.

Shortest distance from Eaton to Madison to Freedom to Eaton is 25 miles.

Distance from Conway to Eaton to Madison to Freedom is 23 miles.


Conway
Madison
Eaton
Freedom



Scale of miles

360

Chapter IV

ARRAYS AND MULTIPLICATION

Background

Arrays. Let us think about 5 objects arranged side by side in a *row*, as suggested below:

x x x x x

With 10 objects, we could think of them as arranged in 2 rows of 5 objects each:

1st row: x x x x x
2nd row: x x x x x

Or, if we had 15 objects, we could think of them as arranged in 3 rows of 5 objects each:

1st row: x x x x x
2nd row: x x x x x
3rd row: x x x x x

And so on.

Any such rectangular arrangement of objects into rows, each containing the same number of objects, is called an *array*. There could, of course, be other kinds of regular arrangements of objects, such as the triangular one suggested below:

x
x x
x x x
x x x x

We may call such an arrangement a "triangular array." However, we use the term "array", without an adjective, when we refer to *rectangular* arrangements, only.

The objects in an array are called its *elements* or *members*. We usually label the rows in an array "1st row", "2nd row", etc., proceeding from top to bottom, as in the illustrations above. In a similar fashion we label the vertical columns in an array "1st column", "2nd column", etc., proceeding from left to right. However, in describing arrays (particularly in connection with the application to multiplication which will follow) we tend to avoid the "column" terminology. Thus, instead of an "array of 3 rows and 5 columns", we speak of an "array of 3 rows of 5 elements each" or, more briefly, of a "3 by 5 array".

It is worth mentioning that a 1 by 5 array, or a 5 by 1 array, or even a 1 by 1 array, are all perfectly legitimate. That is, we permit an array to have only one row, or only one column, or both. Because it will prove helpful in understanding products whose first factor is 0 (such as 0×3, for instance), we also permit the number of rows in an array to be 0. In this case, no matter how many elements we think of each row as having, the number of elements in the array is 0 (since there are no rows at all!). Similarly, because it will be helpful in understanding products whose second factor is 0 (like 5×0, for instance), we permit the number of elements in each row of an array to be 0. In this case, no matter how many rows we think of the array as having, the number of elements in the array is again 0 (since each row has no elements at all!)

Multiplication. The main reason we have introduced arrays at this point is that they are very helpful in defining multiplication and in understanding its properties. Like addition, multiplication is an *operation* in the set of whole numbers. Given a first whole number, say 3, and a second whole number, say 4, the operation of addition performed on these numbers yields a whole-number *sum*, denoted

$$3 + 4.$$

Likewise, the operation of *multiplication* performed on these same numbers yields a whole-number *product*, denoted

$$3 \times 4.$$

The numbers 3 and 4 are *addends* of the sum $3 + 4$.
The numbers 3 and 4 are *factors* of the product 3×4.

The operation of addition was defined with the help of sets. To define the sum $3 + 4$, we took a set of 3 members and joined to it a disjoint set of 4 members; the sum $3 + 4$ was then defined as the number of members in the union of these sets.

We now define the operation of multiplication with the help of arrays. To define the product 3×4, for instance, we consider an array consisting of 3 rows of 4 elements each; the product 3×4 is then defined as the number of elements in this 3 by 4 array.

In counting the number of elements in an array consisting of several rows of, say, 4 elements each, we may of course "count by rows". If the array has just 1 row, it has "1 times 4", or 4, elements; if it has 2 rows, it has "2 times 4", or 8, elements; if it has 3 rows, it has "3 times 4", or 12, elements; and so on. That is, the number of rows tells how many times we take 4 as an addend in finding the number of elements in the whole array:

$$\begin{aligned} &1 \times 4 = 4, \\ &2 \times 4 = 4 + 4, \\ &3 \times 4 = 4 + 4 + 4, \\ &4 \times 4 = 4 + 4 + 4 + 4, \\ &\text{etc.} \end{aligned}$$

This is the so-called "repeated addend" approach to multiplication. It is certainly the approach suggested when we read an expression like

$$5 \times 8$$

as "5 times 8" instead of "the product of 5 and 8". But it is important to keep in mind that our definition of multiplication was not in terms of repeated addition but rather directly in terms of arrays.

Multiplication Properties of 1 and 0. We have already noted that there are 1 by 5 arrays, 5 by 1 arrays, and even 1 by 1 arrays. Such arrays yield the multiplication facts

$$1 \times 5 = 5,$$
$$5 \times 1 = 5,$$
$$1 \times 1 = 1.$$

In general, for any whole number n, we have similarly

$$1 \times n = n$$

and

$$n \times 1 = n.$$

We have also noted that for any whole number n, a 0 by n array or a n by 0 array has no elements at all. Thus we see that for any whole number n,

$$0 \times n = 0$$

and

$$n \times 0 = 0.$$

Commutativity of Multiplication. Recall that addition has the commutative property ($3 + 5 = 5 + 3$, etc.). This was easily seen by means of sets. When we take a set of 3 and join to it a disjoint set of 5, the union is the same as it is when we take this set of 5 and join to it the set of 3.

Like addition, multiplication has the commutative property ($3 \times 5 = 5 \times 3$, etc.). This is easily shown by means of arrays. We take a 3 by 5 array

x x x x x
x x x x x
x x x x x

and rotate it so that the rows become columns and the columns and the columns rows:

x x x

x x x

x x x

x x x

x x x

Since the resulting new array clearly has the same number of elements as the original array, this shows that

$$3 \times 5 = 5 \times 3.$$

Using an array pictured on a large piece of cardboard, a teacher can actually carry out this rotation as a vivid class demonstration.

Prime and Composite Numbers. Some numbers can be represented by an array only if the array has just one row or just one element in each row. Such numbers (for example: 2, 3, 5, 7, 7, 23, 59) are called *prime numbers*, or *primes*. Those numbers which can be represented by arrays with more than one row of two or more elements are called *composite numbers*.

Multiples. A *multiple* of a number, n, is the product of a whole number and n. Thus, multiples of 2 are 2, 4, 6, 8, 10, ... Multiples of 3 are 3, 6, 9, 12, 15, ... A prime number is a multiple only of itself and 1. A composite number is the multiple of at least one number smaller than itself and greater than 1. The number 1 is not included in either the set of primes or the set of composite numbers.

IV-1. Arrays

Objective: To extend children's understanding of the array as a physical model for the operation of multiplication.

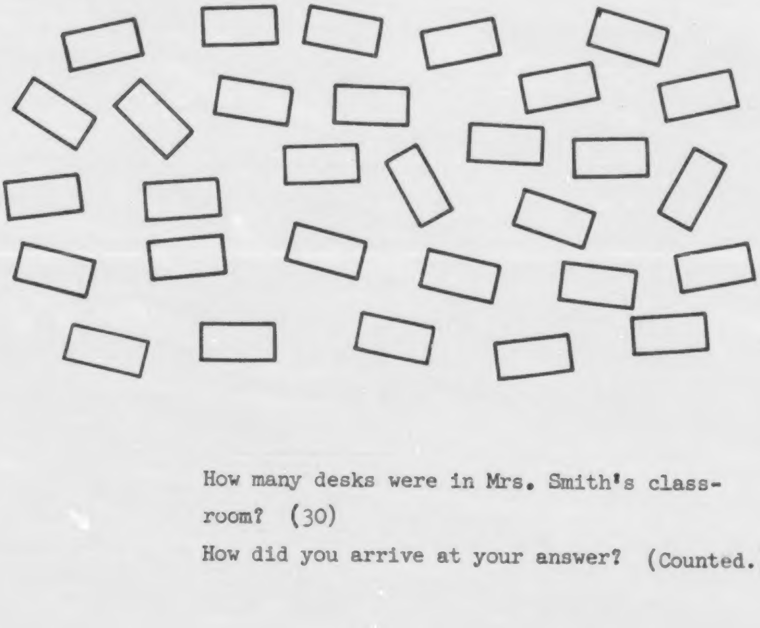
Vocabulary: (Review) Array, row, orderly arrangement.

Materials: Objects for use on the flannelboard.

Suggested Procedure:

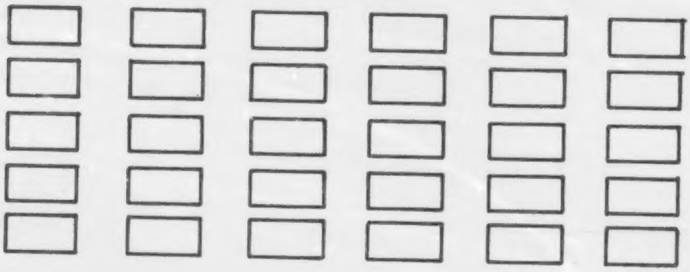
Use flannelboard objects or make drawings on the chalkboard to represent a situation such as the following:

When Mrs. Smith came to school one Monday morning, she found that the janitor had not put the children's desks back in place after he swept the room. The desks were left like this:



How many desks were in Mrs. Smith's classroom? (30)
How did you arrive at your answer? (Counted.)

Since Mrs. Smith was an orderly person, she hurried to rearrange the desks before the children arrived. When she finished the desks were arranged like this.



Was her arrangement of the desks an orderly arrangement? (Yes.)
How many rows are there? (Five.)
How many desks in each row? (Six.)

When we have objects arranged in rows with the same number of objects in each row, we call such an arrangement an array.

Does Mrs. Smith's arrangement of the desks form an array? (Yes.)
Did the janitor's arrangement form an array? (No.)
Why not? (The desks were not arranged in rows with the same number of desks in each row.)

We call Mrs. Smith's arrangement a 5 by 6 array since there are 5 rows with 6 desks in each row.

How many objects are in the array formed by Mrs. Smith's arrangement of the desks? (30.)

Did you have to count to find the number of objects? (No.)

How did you find the number of objects? (6 + 6 + 6 + 6 + 6 or 5 × 6.)

Suppose Mrs. Smith had arranged the desks like this:

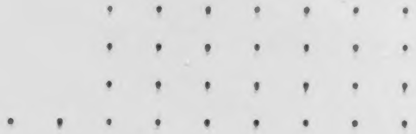
Is this arrangement an array? (Yes.)

How do you know? (There are six rows with five objects in each row.)

What kind of an array would we call this? (6 by 5 array.)

How many objects in this array? (30.)

How did you find the number of objects in this array? (5, 5, 5, 5, 5, and 5 or multiplied 5 by 6.)



Why not? (There are 5 rows but one row has nine desks and the other rows have only seven.)

How would you describe *this* array?

(2 by 4 array.)

How many objects in this array? (Eight.) How can we find the number of members in an array? (Count. Add the number of objects in each row the appropriate number of times, in this case, 4 + 4. Multiply the number of objects in each row by the number of rows, in this case, 2 × 4.)

Pupil's book, page 213:

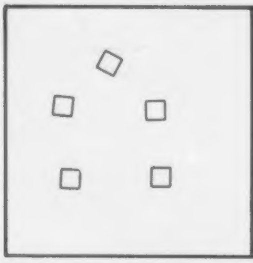
This page requires that the children decide if a set of objects is in the form of an array or not. If not, they are to rearrange the objects into an array. In many cases more than one arrangement is possible.

Page 214:	Drawing of an array that matches the description.
Page 215:	Matching arrays with equations.

Arrays

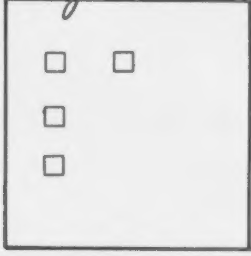
In the pictures below rearrange the objects to form an array. Write in the blanks the number of rows in your array and the number of objects in each row.

Answers will vary. Some possible answers are given.

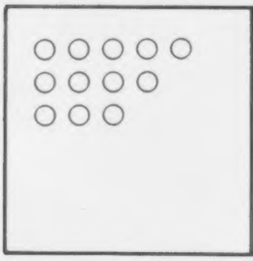


□□□□□

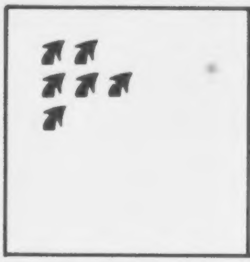
or 5 by 1
1 by 5



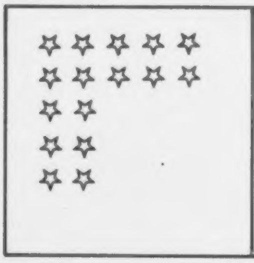
4 by 1
or 1 by 4
or 2 by 2



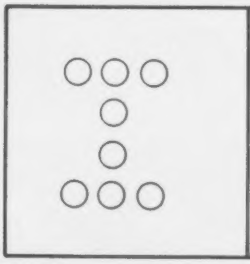
2 by 6
6 by 2
4 by 3
3 by 4



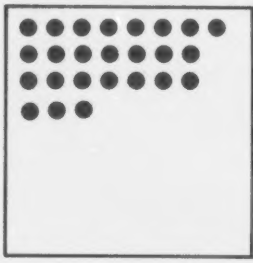
2 by 3
3 by 2



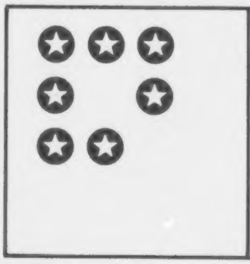
8 by 2
2 by 8
4 by 4



4 by 2
2 by 4



5 by 5



1 by 7
7 by 1

The Number of Elements in an Array

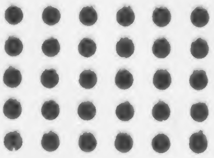
Draw an array, then fill in the blank.

X X X X X X X X X X X X X X X A 5 by 3 array has 15 elements.	X X X X X X X X X X X X X X X X A 4 by 4 array has 16 elements.
X X X X X X X X X X X X X X X X X X X X X A 7 by 3 array has 21 elements.	X X X X X X X X X X X X X X X X X X X X X X X X A 4 by 6 array has 24 elements.
X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X A 4 by 9 array has 36 elements.	X X X X X X X X X X X X X X X X X X X X X X X X An 8 by 3 array has 24 elements.
X X X X X X X X X X X X X X X X X X A 3 by 6 array has 18 elements.	X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X X An 8 by 5 array has 40 elements.

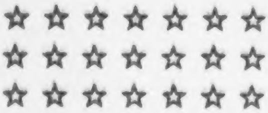
Arrays and Equations

Match the array with the equation that describes it.

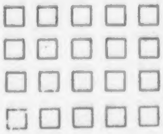
A



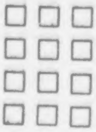
B



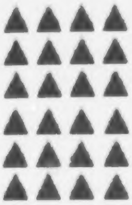
C



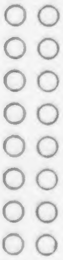
D



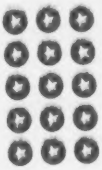
E



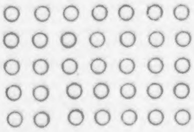
F



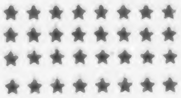
G



H



I



1) 8 + 8 + 8 + 8 = 32 I

2) 6 × 4 = 24 E

3) 4 × 5 = 20 C

4) 3 + 3 + 3 + 3 = 12 D

5) 5 × 6 = 30 A

6) 5 × 7 = 35 H

7) 7 + 7 + 7 = 21 B

8) 5 × 3 = 15 G

9) 8 × 2 = 16 F

IV-2. Multiplication

Objective: To extend the child's understanding of multiplication.

Vocabulary: (Review) Multiplication, factor, product, times, × (symbol).

Materials: Flannelboard materials for showing arrays.

Suggested Procedure:

Begin with a problem situation, such as:

> Every day for a school week, Mrs. Smith brought 3 books back to the library. How many books did she return that week?

Arrange flannelboard materials to represent the books Mrs. Smith returned to the library.

X X X
X X X
X X X
X X X
X X X

Can this problem be solved using addition? (Yes.)
What equation should be written to describe this problem? ($3 + 3 + 3 + 3 + 3 = 15$)
The materials on the flannelboard form an array.
What kind of an array is it? (5 by 3 array)
We have the number 5 for the number of rows, and the number 3 for the number of elements in each row. What is the equation suggested by this array? ($5 \times 3 = 15$.)

How do we read this multiplication equation?
(5 times 3 equal 15.)

Arrange other arrays on the flannelboard such as those represented below and ask the children to write the multiplication equation suggested by each array.

```
. . . .    . . . . .    . . . . . .
. . . .    . . . . .    . . . . . .  etc.
                        . . . . . .
```

Explain that when we write an addition equation such as: 6 + 4 = 10, the 6 and the 4 are called addends. The 10 is called the sum.

Then explain that in a multiplication equation such as 5 × 3 = 15, 5 and 3 are called factors and 15 is called the product.

Review the equations that have been suggested by the arrays. Ask children to name the product in each and the factors of each product.

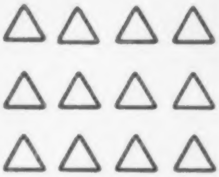
Pupil's book, pages 216-217:

These pages are for independent work.

Multiplication Equations

Fill in the blanks:

1.

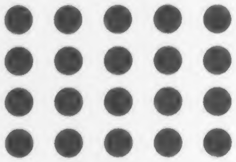


Equation: 3 X 4 = 12

Product: 12

Factors: 3, 4

2.

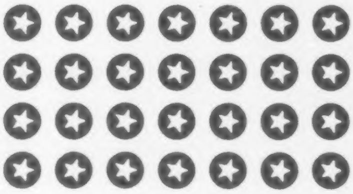


Equation: 4 X 5 = 20

Product: 20

Factors: 4, 5

3.

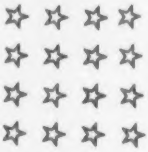


Equation: 4 X 7 = 28

Product: 28

Factors: 4, 7

4.

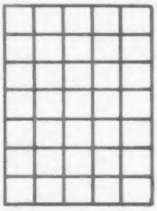


Equation: 4 X 4 = 16

Product: 16

Factors: 4, 4

5.

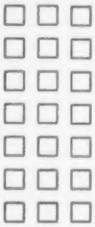


Equation: 7 X 5 = 35

Product: 35

Factors: 7, 5

6.

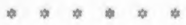


Equation: 7 X 3 = 21

Product: 21

Factors: 7, 3

7.

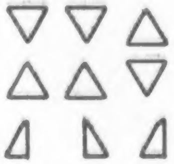


Equation: 1 X 6 = 6

Product: 6

Factors: 1, 6

8.

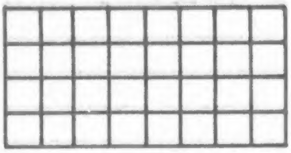


Equation: 3 X 3 = 9

Product: 9

Factors: 3, 3

9.

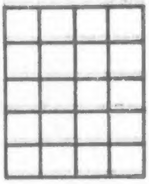


Equation: 4 X 8 = 32

Product: 32

Factors: 4, 8

10.



Equation: 5 X 4 = 20

Product: 20

Factors: 5, 4

11.

a b c d
e f g h
i j k l

Equation: 3 X 4 = 12

Product: 12

Factors: 3, 4

12.

b a d c
f e h g
j i k l

Equation: 3 X 4 = 12

Product: 12

Factors: 3, 4

IV-3. The basic multiplication facts

Objective: To use a table to present the multiplication facts through 9 × 9.

Vocabulary: (No new words.)

Materials: Form for multiplication table drawn on chalkboard or chart.

Suggested Procedure:

Remind the children of their experiences in constructing an addition table. Display the form for the multiplication chart.

Ask a child for the product of 2 and 9. Draw your left hand across the table on the 2 row, and your right hand down on the 9 column to meet it. Record the 18 in the box. Have children come and point to the boxes for recording many other products. (It may be necessary to complete the work in class, and let the children refer to it when working in their books. If there seems to be little difficulty in understanding what is to be done, however, the class-prepared table may be concealed while individual work is completed.)

As the table is being completed, call attention to the way in which the table verifies the commutative property of multiplication. Point out that the product for every pair of factors appears twice in the table, except for the product of a number with itself. With colored chalk, show that the number in row 6, column 2, is the same as that for row 2, column 6, etc.

In completing the chart do not include the zero facts at first.

After chart has been completed with the exception as given above of the zero facts, ask what a product would be if 0 is one of the numbers.

If children are unable to tell what the product would be, motivate their thinking about the problem in the following way.

Show a picture of a 4×6 array. Record equations as you work. Ask how many elements there are in an array with 4 rows of 6 elements each. Write $4 \times 6 = 24$. Cover the bottom row, and ask how many elements there are in an array with 3 rows of 6 elements each. Write $3 \times 6 = 18$ under the earlier equation. Continue covering one row more each time. You will have written

$$\begin{aligned} 4 \times 6 &= 24 \\ 3 \times 6 &= 18 \\ 2 \times 6 &= 12 \\ 1 \times 6 &= 6 \end{aligned}$$

Since it is not possible to show an array of 0 rows with 6 elements in the row (covering all of the rows of an array is a representation of only one zero fact, $0 \times 6 = 0$), children should note that each time you cover one more row the product decreases by 6. This makes it possible to write $0 \times 6 = 0$. In addition, children can be asked to make other arrays including an array which has zero rows and a number of elements in the row. The attempt to show such an array makes it clear that there can be no elements in such an array or that if zero is a factor, the product is zero.

Do the same thing, covering the columns of the array one at a time and writing the equations

$$\begin{aligned} 4 \times 6 &= 24 \\ 4 \times 5 &= 20, \text{ etc.} \end{aligned}$$

Pupil's book, pages 218-220:

These pages are concerned with basic multiplication facts. Perhaps the multiplication table should be completed as a class exercise if children seem able to do it at this time.

A Multiplication Table

Write the product for each pair of factors, for example, 2 × 6 = 12, and 6 × 2 = 12.

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Zero or One as a Factor

Write the products.

5 × 0 = 0

0 × 5 = 0

0 × 0 = 0

0 × 641 = 0

0 × n = 0

1,240 × 0 = 0

1 × 9 = 9

9 × 1 = 9

1 × 89 = 89

1 × 13 = 13

1 × 17 = 17

n × 1 = n

How would you complete these equations?

________ × 0 = 2 Impossible

0 × ________ = 7

If zero is a factor, what is the product? 0

If one is a factor, what is the product? The other factor

Other Factors

1. Start at 0 and count to 18 by 2's.

   0 2 4 6 8 10 12 14 16 18

2. What row in your chart looks like your answer to question 1?

   2 What column? 2

3. Start at 0 and count to 18 by 3's.

   0 3 6 9 12 15 18 ___ ___

4. What row in your chart looks like your answer to question 3?

   3 What column? 3

5. Start at 0 and count to 18 by 4's.

   0 4 8 12 16 ___ ___ ___ ___

6. What row in your chart looks like your answer to question 5?

   4 What column? 4

7. How can you tell just by looking at a product that it has 5 as a factor? The last digit is 0 or 5

8. Why is there a row and a column that look like counting from 0 to 18 by 6's? When you count by 6 you name the product of 6 with each whole number.

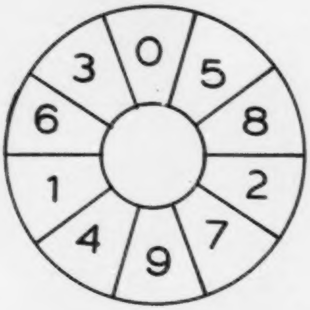
9. How many products appear in your chart only once? 6

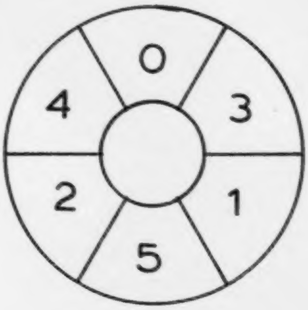
   Why? Answers will vary.

Additional practice:

"Drill Doughnuts" may be used to advantage for additional practice in adding, subtracting, and multiplying both for the basic facts and for encouraging mental computation later (35 + 5, 49 - 6, etc.)

For each child in the class, prepare a Doughnut. Cut from cardboard, tagboard, or other heavy stock. (Circle is $4\frac{1}{4}$ inches in diameter, center hole is 1 inch in diameter.) On one side use red felt pen for the numerals, on the other a blue felt pen.





Give each child a Doughnut and a sheet of newsprint, 9 × 12. Tell children to fold paper in half (to yield two sections on each side of paper, each 9 × 6). Have Doughnut placed on paper so that zero is at the top and there is room on each section of paper to write numerals around the edge of the Doughnut. Tell children to hold Doughnut still, not to trace around it, but to write on paper through the hole in the middle "5 × ". Beyond the circumference of the Doughnut, on the newsprint, they should write products of 5 and the numbers indicated on the sections of the Doughnut.

Next move Doughnut to another section of the paper. Give the operation sign and the number to be written: 4 ×, 7 ×, etc. Children again write answers around the Doughnut on the paper. This is repeated when the paper is turned over.

When all four sections of paper are finished, 40 problems have been done if blue side is used, 24 if red side is used. The teacher can make a key, and since only the center entry and the answers appear for each section, checking papers is easy. Eventually all that is needed in the way of preparation for practice is the following on the chalkboard:

5 x	7+

Red Side

10-	12-

Blue Side

The Doughnut can also be used for division, but in this case the children should be told to put a big X opposite zero since you can't divide by zero.

IV-4. Prime Numbers

Objective: To present the multiplication facts in a new and important setting.

Vocabulary: Multiple, product, prime number.

Background:

A number greater than one is called prime if it is a multiple only of itself and one. The first few prime numbers are:

2, 3, 5, 7, 11, 13, 17, ...

The pattern they form within the set of whole numbers is strangely irregular. Many questions about them remain unanswered to this day. They are nonetheless fundamental in the structure of the number system: every whole number greater than one is either a prime number or a unique product of prime numbers.

Much of the individual character of a number is determined by its composition as a product of primes. Twelve, for example, is famous as a highly divisible number. This is a result of its being composed of three small prime factors: $12 = 2 \times 2 \times 3$. Thirteen, by contrast, is notorious for its indivisibility. Prime numbers have always seemed mysterious. Most of the small ones, notably 3, 7, 11, and 13, are prominent in superstitions and fairy tales. A feeling for the individual character of numbers should be helpful in learning the multiplication facts. Later on, prime factors will be used in connection with division and names for a rational number.

Suggested Procedure:

Ask a child to try to make an array with 23 objects. Convince the class that the only array possible is a 1×23 (or 23×1) array. Explain that 23 is called a prime number because of this property. Let them test a few other numbers. Pupil pages 221-222 lead the child to test the first twenty numbers systematically. You may wish to go farther than this in classroom discussion. Pupil pages 223-225 present an alternative method of generating the prime numbers. In a classroom discussion let the children count the different arrays that can be made with a given number of objects. Lead them to see how the number of arrays is related to the number of prime factors.

* Pupil's book, page 226-229. (Optional) These pages lead more able pupils to discover some interesting characteristics of triangle numbers and square numbers.

* Pupil's book, pages 230-234. (Optional) More able pupils may explore some relationships between multiplication and addition.

Prime Numbers and Products of Primes

Suppose you want to arrange a set of objects in an array. You can always make an array with just one row like this:

0 0 0 0 0 0 0 0 0

or just one object in each row like this:

0
0
0
0
0
0
0
0
0

But can you always make an array with more than one row and more than one object in each row? Let's see. Can you do it with 12 objects? Yes.

If you can, draw the array here:

0000 000000 or 4 by 3 or 6 by 2
0000 000000
0000

Can you do it with 9 objects? Yes. If you can, draw the array here:

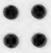
000
000
000

Can you do it with 11 objects? No. If you can, draw the array here:

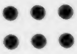
Now try it for all the numbers listed below. For each number try to make an array with more than one row and more than one object in each row. If you can do it, draw the array. If you can't do it, put an X in the blank by the number.

2 X

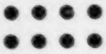
3 X

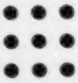
4 ____ 

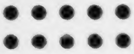
5 X

6 ____ 

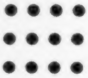
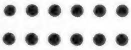
7 X

8 ____ 

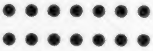
9 ____ 

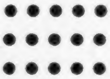
10 ____ 

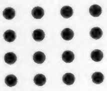
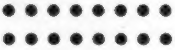
11 X

12 ____  

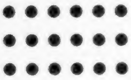
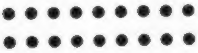
13 X

14 ____ 

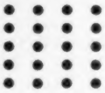
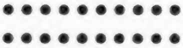
15 ____ 

16 ____  

17 X

18 ____  

19 X

20 ____  

The numbers you have marked with X are called prime numbers. As you go higher, the prime numbers get scarcer, but no matter how high you go there are always more prime numbers farther on. The set of prime numbers forms a mysterious and irregular-looking pattern.

These numbers are called multiples of 2:

2, 4, 6, 8, 10, 12, ...

These numbers are called multiples of 3:

3, 6, 9, 12, 15, 18, 21, ...

Write here the first ten multiples of 5:

0, 5, 10, 15, 20, 25, 30, 35, 40, 45 (50, if 0 is omitted

What special name do we have for the multiples of 2? Even numbers.

Every number is a multiple of 1, and every number is a multiple of itself.

In the list below put a 1 next to every multiple of 1, put a 2 next to every multiple of 2, put a 3 next to every multiple of 3, and so forth as far as you can go.

2	1	2				prime	10	1	2	5	10		
3	1	3				prime	11	1	11				prime
4	1	2	4				12	1	2	3	4	6	12
5	1	5				prime	13	1	13				prime
6	1	2	3	6			14	1	2	7	14		
7	1	7				prime	15	1	3	5	15		
8	1	2	4	8			16	1	2	4	8	16	
9	1	3	9				17	1	17				prime

18 1 2 3 6 9 18

19 1 19 ___ ___ prime

20 1 2 4 5 10 20

21 1 3 7 21 ___ ___

. . .

(The three dots after 21 show that you could go on and on.)

How many numbers is 8 a multiple of? 4

What is the smallest number that is a multiple of six numbers? 12

What is the smallest number that is a multiple of exactly five numbers? 16

Write Prime next to each number that is a multiple of no number except itself and 1. Does this check with the prime numbers you found using arrays? Yes If not, go back and check your work.

Do you remember what product means? What is the product of 2 and 5? 10

Every whole number greater than 1 is either a prime number or can be written as a product of prime numbers. Write each of the following numbers as a product of prime numbers. Be careful to use only prime numbers. Some of them are done for you.

2 prime

3 prime

4 = 2 × 2

5 prime

6 = 2 × 3

7 prime

8 = 2 × 2 × 2

9 = 3 X 3

10 = 2 X 5

11 prime

12 = 2 X 2 X 3

13 prime

14 = 2 X 7

15 = 3 X 5

16 = 2 X 2 X 2 X 2

17 prime

18 = 2 X 3 X 3

19 prime

20 = 2 X 2 X 5

21 = 3 X 7

22 = 2 X 11

23 prime

24 = 2 X 2 X 2 X 3

25 = 5 X 5

26 = 2 X 13

27 = 3 X 3 X 3

28 = 2 X 2 X 7

29 prime

30 = 2 X 3 X 5

31 prime

Is this equation correct? Yes

$$2 \times 3 \times 5 = 30$$

How many numbers is 30 a multiple of? 8 {1, 2, 3, 5, 6, 10, 15, 30}

How many different arrays could you make with 30 objects? 4 (If you count 5 X 6 the same as 6 X 5, etc.)

How many different arrays could you make with 100 objects? 5

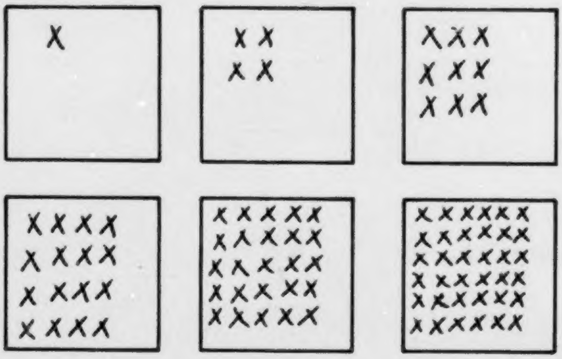
1 X 100
2 X 50
5 X 20
4 X 25
10 X 10

(It helps to write 100 as a product of primes:
100 = 2 X 2 X 5 X 5)

★ Square and Triangular Arrays.

1. 3 × 3 is sometimes called the "square of 3." Can you think why? It can be represented by a square array.
Write the squares of the first six numbers and draw an array for each one.

1 × 1 = 1
2 × 2 = 4
3 × 3 = 9
4 × 4 = 16
5 × 5 = 25
6 × 6 = 36



2. Now do these additions:

1 = 1
1 + 3 = 4
1 + 3 + 5 = 9
1 + 3 + 5 + 7 = 16
1 + 3 + 5 + 7 + 9 = 25
1 + 3 + 5 + 7 + 9 + 11 = 36

3. Compare the answers you got in problems 1 and 2. What do you notice? They are the same.

4. Here is a dot.

●

Make a 2 × 2 array by putting in more dots. How many more dots did you have to put in? 3

Now make it into a 3 × 3 array. How many more dots did you need? 5

Now make it into a 4 × 4 array. How many more dots did you need? 7

Now make it into a 5 × 5 array. How many more dots did you need? 9

Now make it into a 6 × 6 array. How many more dots did you need? 11

5. Now look back at problems 1 and 2. Can you explain, using what you found out in problem 4, why you got the same answers to both problems 1 and 2? Answers will vary.

6. The numbers 1, 4, 9, 16, 25, 36, ... etc., are called the square numbers. They are the numbers of things in square arrays. There is another set of numbers called the triangle numbers. These are the numbers of things in triangular arrays. Here are the first few triangle numbers with their arrays:

o 1

o
o o 3

o
o o
o o o 6

o
o o
o o o
o o o o 10

7. Do these additions:

1 = 1

1 + 2 = 3

1 + 2 + 3 = 6

1 + 2 + 3 + 4 = 10

1 + 2 + 3 + 4 + 5 = 15

1 + 2 + 3 + 4 + 5 + 6 = 21

Did you get the triangle numbers? Yes

Explain why. The rows of the triangular arrays have 1, 2, 3, . . etc. objects.

8. Here are the first few triangle numbers:

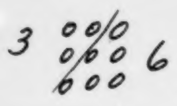
1, 3, 6, 10, 15, 21, 28, ...

Let's add them in pairs.

1 + 3 = 4
3 + 6 = 9
6 + 10 = 16
10 + 15 = 25
15 + 21 = 36
21 + 28 = 49

What numbers did you get? Square numbers. Can you explain why? Pairs of triangular arrays can be fitted together to make square arrays, as shown below.

Hint: Try to fit two triangular arrays together.



★ Multiplying and Adding

1. Here are two sets of numbers:

Set A: 2, 3, 5

Set B: 4, 6

Write down all the pairs of numbers you can make taking the first number from Set A and the second from Set B.

2, 4
2, 6
3, 4
3, 6
5, 4
5, 6

We can show the set of number pairs you have just written by means of an array:

		Set B	
		4	6
Set A	2	.	.
	3	.	.
	5	.	.

Each dot in the array stands for one of the possible number pairs. Compare your list of pairs with the array. Do they check? Yes

2. Multiply each pair of numbers in your list and put the product into this array. The product of 6 and 3 has been put in for you to show you where it goes.

		Set B	
		4	6
Set A	2	8	12
	3	12	18
	5	20	30

3. Add up the six numbers inside the array and put their sum in this box:

$\boxed{100}$

We will come back to this number.

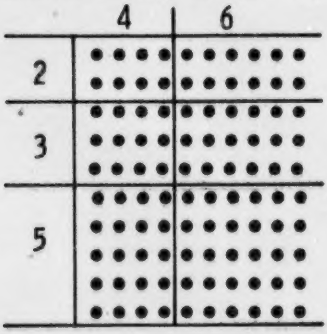
4. Find the sum of the numbers in Set A: 2 + 3 + 5 = 10

And in Set B: 4 + 6 = 10

5. Multiply these two sums together and put the product in this box:

$\boxed{100}$

6. Now look at the numbers you have in the two boxes (problems 3 and 5). Are they the same? If they are not, go back and check your work. The two numbers should be the same. To see why, look at this array.



How many dots are there in each of the rectangular pieces of the array?
8, 12, 12, 18, 20, 30

How many dots are there in the whole array? 100.

Now explain why you got the same number in problems 3 and 5.

To get the number of dots in the whole array, you can add the numbers of dots in the rectangular pieces of the array or you can multiply the number of rows by the number of dots in each row.

7. Fill in this array with the products of the numbers in Set A with those in Set B. One product has been put in for you.

Set B

		1	4	3
	2	2	8	6
Set A	5	5	20	15
	3	3	12	9

What is the sum of the numbers inside the array? 80

Could you have found this out without actually filling in the array? Yes How? By multiplying 1+4+3, or 8, by 2+5+3, or 10.

8. What is 13 × 13? 169

Here is a way to find 13 × 13 using what we have learned. Fill in this array with the products as before:

	10	3
10	100	30
3	30	9

What is the sum of the numbers you put in the array? 169

Is this sum equal to 13 × 13? Yes

Why? The sum of the numbers in the array is equal to 10 + 3 times 10 + 3.

9. Use arrays to find these products:

11 × 11 = 121

12 × 12 = 144

14 × 14 = 196

15 × 15 = 225

Array for 15 × 15

	10	5
10	100	50
5	50	25

10. When we write

$$(2 + 3) \times (4 + 5) = \_\_\_\_$$

we mean that you must first do the additons inside the parentheses to get

$$5 \times 9$$

and then do the multiplication to get 45.

When we write

$$(2 \times 3) + (4 \times 5)$$

we mean that you must first do the multiplications inside the parentheses to get

$$6 + 20$$

and then do the addition to get 26.

Always do what is inside the parentheses first.

Is this equation correct? Do the arithmetic to find out. Yes.

(2 + 3) × (2 + 5) = (2 × 2) + (2 × 5) + (3 × 2) + (3 × 5)
5 x 7 = 4 + 10 + 6 + 15

Can you make a product array to go with this equation?

Explain what the equation says about the array.

	2	3
2	4	6
5	10	15

or

	2	5
2	4	10
3	6	15

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